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Tone Reservation for OFDM with Restricted Carrier Set

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***Abstract*—The tone reservation method can be used to reduce the peak to average power ratio (PAPR) in orthogonal frequency division multiplexing (OFDM) transmission systems. In this paper the tone reservation method is analyzed for OFDM with a restricted carrier set, where only the positive carrier frequencies are used. Performing a fully analytical analysis, we give a complete characterization of the information sets for which the PAPR problem is solvable. To derive our main result, we connect the PAPR problem with a geometric functional analytic property of certain spaces. Further, we present applications of our theory that give guidelines for choosing the information carriers in the finite setting and discuss a probabilistic approach for selecting the carriers. Additionally, it is shown that if there exists an information sequence for which the PAPR problem is not solvable, then the set of information sequences for which the PAPR problem is not solvable is a residual set.**

***Index Terms*—Peak to average power, crest factor, tone reservation, orthogonal frequency division multiplexing, solvability**

I. Introduction

ORTHOGONAL transmission schemes, like orthogonal frequency division multiplexing (OFDM), are popular in modern communication systems. Mathematically, the transmit signal s of an orthogonal transmission scheme has the form

$$s(t) = \sum_{k\in\mathcal{I}} c_k \phi_k(t), \quad t \in [t_1, t_2],$$

where $\mathcal{I} \subset \mathbb{Z}$ is an index set, $t_2 - t_1$ is the signal duration, $\{\phi_k\}_{k\in\mathcal{I}}$ is an orthonormal system (ONS) of functions on $[t_1, t_2]$, and $\{c_k\}_{k\in\mathcal{I}} \subset \mathbb{C}$ are the information bearing coefficients. Each ϕ_k is called carrier. In modern standards, such as 5G, also arbitrary waveforms $\{\phi_k\}_{k\in\mathcal{I}}$, which are not necessarily orthogonal, are discussed [2].

Although orthogonal transmission schemes have many favorable properties [3], they suffer from large peak to average power ratios (PAPRs) of the transmit signals [4], [5]. For any orthonormal system $\{\phi_k\}_{k\in\mathcal{I}}$ with bounded functions ϕ_k, $k \in \mathcal{I}$, the worst-case PAPR increases like $\sqrt{N}$, where N denotes the number of carriers. Large PAPRs are problematic, because they can overload power amplifiers, leading to distorted signals and out-of-band radiation [6].

Numerous methods have been proposed for reducing the PAPR [7]–[23]. In addition to selected mapping [13], [15], [17], [18], [24], the method of tone reservation, which we consider in this paper, is often employed [7], [8], [11], [20], [25], [26]. In the tone reservation method, the set of available carriers $\{\phi_k\}_{k\in\mathcal{I}}$ is partitioned into two sets, the first of which is used to carry the information, and the second of which to reduce the PAPR.

The tone reservation method has been extensively studied for OFDM systems [11], [12], [16]. Although tone reservation is an elegant method that is easy to describe, its theoretical analysis is difficult. Almost all studies so far are based on numerical approaches, and there exist virtually no analytical results for the solvability of the PAPR problem. It is clear that brute force numerical approaches are problematic due to the combinatorial nature of the problem, which often makes it infeasible to find optimal solutions.

An analytical study of the PAPR reduction in OFDM with tone reservation was performed in [27], where Kashin's representation was used to construct a transmit signal, the PAPR of which is bounded from above by a certain expression. Further, a PAPR reduction algorithm was presented. A different line of research has been conducted in [9], [28], [29]. There the PAPR of OFDM for certain codes was studied in connection with the capacity of the channel. In [9] a lower bound on the PAPR for constant energy codes was derived, and in [28] more general codes were considered.

In practical applications the following three questions for tone reservation for orthonormal transmission schemes are relevant: What is the best possible reduction of the PAPR? What is the optimal information set that achieves this reduction, and how can it be found? What is the general structure of the information set? The answers to these questions are difficult to obtain and open, except for special cases. For CDMA communication systems that employ the Walsh system, the question about the best possible reduction constant could be answered in [30], and an optimal information set that achieves this reduction was given. For OFDM the answers to these questions are, to the best of our knowledge, unknown.

A starting point for the analysis of the PAPR problem for general complete orthonormal systems (CONS) was made in [31]. There it has been analyzed when, for a given CONS, the PAPR reduction problem is solvable, and two solution concepts—weak and strong solvability—were introduced. We will discuss both concepts later in more detail. For the results

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in [31] it has been important that the employed ONS was complete. This property was central at several points in the proof. It seems that this assumption cannot be weakened without loosing the generality of the result. If the completeness is not given, specific properties of the ONS at hand have to be utilized to obtain comparable results.

In this paper we will study the PAPR problem for an orthonormal system that is not complete. This requires a completely different and new approach compared to [31]. In particular the result for the restricted carrier set cannot be obtained by a projection of the unrestricted carrier set.

Before trying to answer the above three difficult questions, it seems reasonable to address a more fundamental question first: When is the PAPR reduction problem solvable? In [25] and [32] an analytical approach was taken to answer this question. An essential assumption in the analysis was that the set of carrier functions $\{\phi_k\}_{k\in\mathcal{I}}$ forms a complete orthonormal system. However, for practical applications, this assumption may be too restrictive. For example in OFDM, usually only a subset of all carrier frequencies is used. In this case the system of carrier functions is no longer complete and the theory and proof ideas from [32] are no longer applicable.

Given an orthonormal system $\{\phi_k\}_{k\in\mathcal{I}}$, the information set and the compensation set both have to be subsets of the index set $\mathcal{I}$. In OFDM the employed system of orthonormal functions is $\{\mathrm{e}^{ikt}\}_{k\in\mathcal{I}}$, and only a fraction of all available carriers is used, i.e., we have $\mathcal{I}\subset\mathbb{Z}$. In the present work we analytically analyze the PAPR problem for OFDM with an index set $\mathcal{I}=\mathbb{N}_0$, and completely characterize the information sets for which the PAPR is solvable. To the best of our knowledge, there exist no analytical results for these problems so far. Note that our analytical results are formulated in an infinite setting, where we allow infinitely many compensation carriers. Clearly, in practice only finitely many carriers can be used, which reduces the performance.

We further present applications of our theory that give guidelines for choosing the information carriers. By using the theory of arithmetic progressions, we can derive results for the practical relevant case of finitely many carriers, and for a probabilistic choice of the information carriers. We will see that the density of the information carriers is an important quantity in these considerations.

II. Notation

By $\mathbb{Z}$ we denote the integers, by $\mathbb{N}$ the naturals, and by $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ the non-negative integers including zero. For a set $\mathcal{K}\subset\mathbb{Z}$ we use the notation $-\mathcal{K}=\{-k\colon k\in\mathcal{K}\}$. Further, by $L^p[-\pi,\pi]$, $1\leq p\leq\infty$, we denote the usual L^p-spaces on the interval $[-\pi,\pi]$, equipped with the norm $\|f\|_{L^p}=(1/(2\pi)\int_{-\pi}^{\pi}|f(t)|^p\,\mathrm{d}t)^{1/p}$ for $1\leq p<\infty$ and with $\|f\|_{L^\infty}=\operatorname{ess\,sup}_{t\in[-\pi,\pi]}|f(t)|$ for $p=\infty$. ess sup denotes the essential supremum. Note that for notational convenience we use a normalization factor in the norm. We can identify functions defined on $[-\pi,\pi]$ with functions defined on the torus $\mathbb{T}$. μ denotes the Lebesgue measure. For an index set $\mathcal{I}\subset\mathbb{Z}$, we denote by $\ell^2(\mathcal{I})$ the set of all square summable sequences $c=\{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$. The norm is given by $\|c\|_{\ell^2(\mathcal{I})}=(\sum_{k\in\mathcal{I}}|c_k|^2)^{1/2}$.

III. PAPR, Tone Reservation and Solvability Concepts

A. Basic Concepts

Without loss of generality, we restrict ourselves to signals defined on the interval $[-\pi,\pi]$, i.e., signals with a duration of 2π. Signals with other duration can be simply scaled to be concentrated on $[-\pi,\pi]$. For a signal $s\in L^2[-\pi,\pi]$, we define the crest factor by

$$\mathrm{CF}(s)=\frac{\|s\|_{L^\infty}}{\|s\|_{L^2}},\tag{1}$$

i.e., the CF is the ratio between the peak value of the signal and the square root of the power of the signal. Note that the PAPR is defined as the square of this value. Hence, minimizing the PAPR of a signal is equivalent to minimizing the CF. In the following we will call the PAPR problem CF problem for consistency.

In the case of an orthogonal transmission scheme, using an ONS $\{\phi_k\}_{k\in\mathcal{I}}\subset L^2[-\pi,\pi]$, the CF of the transmit signal

$$s(t)=\sum_{k\in\mathcal{I}}c_k\phi_k(t),\quad t\in[-\pi,\pi],$$

with coefficients $c=\{c_k\}_{k\in\mathcal{I}}$, is given by

$$\mathrm{CF}(s)=\frac{\|\sum_{k\in\mathcal{I}}c_k\phi_k\|_{L^\infty}}{\|c\|_{\ell^2(\mathcal{I})}},$$

because $\{\phi_k\}_{k\in\mathcal{I}}$ is a ONS, and thus $\|s\|_{L^2}=\|c\|_{\ell^2(\mathcal{I})}$.

Remark 1. For an orthogonal transmission scheme, the peak value of the signal s, and hence the CF, can become large, as the following result shows. Given any system $\{\phi_k\}_{k=1}^N$ of N orthonormal functions in $L^2[-\pi,\pi]$, then there exist a sequence $\{c_k\}_{k=1}^N\subset\mathbb{C}$ of coefficients with $\sum_{k=1}^N|c_k|^2=1$, such that $\|\sum_{k=1}^N c_k\phi_k\|_{L^\infty}\geq\sqrt{N}$ [33]. This increase of the CF with an order of $\sqrt{N}$ is undesired and ways to battle it are needed.

Tone reservation is one approach to reduce the CF. Let $\{\phi_k\}_{k\in\mathcal{I}}$ be an ONS in $L^2[-\pi,\pi]$. We additionally assume that $\|\phi_k\|_{L^\infty}<\infty$, $k\in\mathcal{I}$, i.e., we consider the practically relevant case of bounded carriers. In the tone reservation method, the index set $\mathcal{I}$ is partitioned in two disjoint sets, the information set $\mathcal{K}$ and the compensation set $\mathcal{K}_{\mathcal{I}}^{\complement}=\mathcal{I}\setminus\mathcal{K}$. Although slightly imprecise, we will call the carrier index set $\mathcal{I}$ simply carrier set in the following. Note that the set $\mathcal{K}$ can be finite or infinite. For a given sequence $a=\{a_k\}_{k\in\mathcal{K}}\in\ell^2(\mathcal{K})$, the goal is to find a sequence $b=\{b_k\}_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}\in\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})$ such that the peak value of the signal

$$s(t)=\underbrace{\sum_{k\in\mathcal{K}}a_k\phi_k(t)}_{=:A(t)}+\underbrace{\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}b_k\phi_k(t)}_{=:B(t)},\quad t\in[-\pi,\pi],$$

is as small as possible. $A(t)$ denotes the signal part which contains the information and $B(t)$ the part which is used to reduce the CF.

B. Basic Definitions

Note that we allow infinitely many carriers to be used in our theory. In particular, we allow infinitely many carriers to be used for the compensation of the CF. Clearly, this setting is not practical, however, from our results we can derive recommendations for the choice of the information set in the finite setting. This will be discussed in Section VI. Further, the solvability of the CF problem in the infinite setting is a necessary condition for the solvability of the CF problem in the finite setting. Hence, if the CF problem is not solvable for a certain information set $\mathcal{K}$ in the infinite setting it is even more so not solvable with a finite compensation set. This helps us to rule out certain bad information sets, and thus we can reduce the number of information sets which have to be considered in the search for good information sets.

We define the strong solvability of the CF problem next.

Definition 1 (Strong Solvability of the CF problem)**.** For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2[-\pi,\pi]$ and a set $\mathcal{K}\subset\mathcal{I}$, we say that the CF problem is strongly solvable with finite extension constant $C_{\text{Ex}}^{\mathcal{I}}$, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})$ such that

$$\left\|\sum_{k\in\mathcal{K}} a_k\phi_k+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}} b_k\phi_k\right\|_{L^\infty}\leq C_{\text{Ex}}^{\mathcal{I}}\|a\|_{\ell^2(\mathcal{K})}. \tag{2}$$

We call the CF problem strongly solvable if it is strongly solvable for some finite extension constant $C_{\text{Ex}}^{\mathcal{I}}$. The smallest of all extension constants $C_{\text{Ex}}^{\mathcal{I}}$, such that (2) is satisfied, is denoted by $\underline{C}_{\text{Ex}}^{\mathcal{I}}$.

If the CF reduction problem is strongly solvable, condition (2) immediately implies that $\|b\|_{\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})}\leq C_{\text{Ex}}^{\mathcal{I}}\|a\|_{\ell^2(\mathcal{K})}$, because

$$\begin{aligned}\left(\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}|b_k|^2\right)^{\frac12}&\leq\left(\sum_{k\in\mathcal{K}}|a_k|^2+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}|b_k|^2\right)^{\frac12}\\&=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k\in\mathcal{K}}a_k\phi_k(t)+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}b_k\phi_k(t)\right|^2\mathrm{d}t\right)^{\frac12}\\&\leq\operatorname*{ess\,sup}_{t\in[-\pi,\pi]}\left|\sum_{k\in\mathcal{K}}a_k\phi_k(t)+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}b_k\phi_k(t)\right|.\end{aligned} \tag{3}$$

That is, the energy of the compensation signal is bounded by $(C_{\text{Ex}}^{\mathcal{I}}\|a\|_{\ell^2(\mathcal{K})})^2$. Further, we have $\mathrm{CF}(s)\leq C_{\text{Ex}}^{\mathcal{I}}$.

An ONS $\{\phi_k\}_{k\in\mathcal{I}}$ is said to be complete if $\overline{\operatorname{span}(\{\phi_k\}_{k\in\mathcal{I}})}=L^2[-\pi,\pi]$, i.e., if the closed linear span of $\{\phi_k\}_{k\in\mathcal{I}}$ equals $L^2[-\pi,\pi]$, or in other words if the linear span of $\{\phi_k\}_{k\in\mathcal{I}}$ is dense in $L^2[-\pi,\pi]$. In [20], [26] the following characterization of strong solvability of the CF problem was given for ONSs that are complete. In this characterization the set

$$\mathfrak{P}^1(\mathcal{K})=\left\{f\in L^1[-\pi,\pi]\colon\right.\\ \left. f=\sum_{k\in\mathcal{K}}a_k\phi_k\text{ for some }\{a_k\}_{k\in\mathcal{K}}\subset\mathbb{C}\right\} \tag{4}$$

plays a central role. $\mathfrak{P}^1(\mathcal{K})$ is a closed subset of $L^1[-\pi,\pi]$.

Theorem 1. *Let $\{\phi_k\}_{k\in\mathcal{I}}$ be a complete ONS in $L^2[-\pi,\pi]$, $\mathcal{K}\subset\mathcal{I}$, and $C_{\text{Ex}}^{\mathcal{I}}>0$. The CF problem is strongly solvable for $\{\phi_k\}_{k\in\mathcal{I}}$ and $\mathcal{K}$ with constant $C_{\text{Ex}}^{\mathcal{I}}$ if and only if*

$$\|f\|_{L^2}\leq C_{\text{Ex}}^{\mathcal{I}}\|f\|_{L^1} \tag{5}$$

for all $f\in\mathfrak{P}^1(\mathcal{K})$.

Theorem 1 connects the CF problem with a geometric functional analytical property of the set $\mathfrak{P}^1(\mathcal{K})$. Comparing (5) with (2), we see that the optimal extension constant $\underline{C}_{\text{Ex}}^{\mathcal{I}}$ is even the same. In Section VI, we will use the geometric condition (5) to derive a necessary condition for the solvability of the OFDM CF problem and to establish connections to combinatorics. Theorem 1 is particularly interesting, because the characterization only depends on the information set $\mathcal{K}$ and the signals created by this set. That is, the compensation signals are irrelevant.

Remark 2. Due to the Cauchy–Schwarz inequality we have $\|f\|_{L^1}\leq\|f\|_{L^2}$. Hence, if the CF problem is strongly solvable, we have

$$\frac{1}{C_{\text{Ex}}^{\mathcal{I}}}\|f\|_{L^2}\leq\|f\|_{L^1}\leq\|f\|_{L^2}$$

for all $f\in\mathfrak{P}^1(\mathcal{K})$. This shows that on $\mathfrak{P}^1(\mathcal{K})$, the L^1-norm and the L^2-norm are equivalent.

A weaker form of solvability is weak solvability.

Definition 2 (Weak solvability of the CF problem)**.** For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2[-\pi,\pi]$ and a set $\mathcal{K}\subset\mathcal{I}$, we say that the CF problem is weakly solvable, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})$ such that

$$\left\|\sum_{k\in\mathcal{K}} a_k\phi_k+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}} b_k\phi_k\right\|_{L^\infty}<\infty. \tag{6}$$

This is a weaker form of solvability compared to strong solvability, because the peak value of the transmit signal is only required to be bounded and not to be controlled by the norm of the sequence $a=\{a_k\}_{k\in\mathcal{K}}$ as in (2).

C. Discussion

Finding optimal information sets $\mathcal{K}$ and determining the minimal extension constant is very challenging. For general ONSs $\{\phi_k\}_{k\in\mathcal{I}}$ the answers to both questions are unknown.

For the special case where the ONS is given by the set of Walsh functions, answers could be obtained [34]. Walsh functions are used for example in code division multiple access (CDMA) transmission schemes. In [34] it was shown that if more than $N\geq 2$ carriers are used, then the optimal extension constant $\underline{C}_{\text{Ex}}$ is independent of N and given by $\underline{C}_{\text{Ex}}(N)=\sqrt{2}$. Further, it was proved that this minimal extension constant can be achieved by using the first N Rademacher functions, i.e., the information set $\mathcal{K}=\{2^k+1\}_{k=0}^{N-1}$.

The proof techniques that were used in [34] were specifically tailored to the Walsh function and therefore do not work for the system of exponentials that is used in OFDM, or for other ONS.

IV. Solvability for OFDM With Restricted Carrier Set

In Theorem 1 we have seen a functional analytic description of the solvability of the CF problem for the case where the ONS $\{\phi_k\}_{k\in\mathcal{I}}$ is complete. The goal of this section is to study the CF problem for OFDM with a restricted carrier set, which corresponds to a ONS that is not complete.

We are only interested in OFDM, i.e., the employed set of orthonormal functions will be the set of complex exponentials $\{\mathrm{e}^{ikt}\}_{k\in\mathcal{I}}$ in the following. As carrier set we consider $\mathcal{I}=\mathbb{N}_0$, i.e., the case where only the positive frequencies are used. If $\mathcal{I}=\mathbb{Z}$ then $\{\mathrm{e}^{ikt}\}_{k\in\mathbb{Z}}$ is a complete ONS in $L^2[-\pi,\pi]$. For this case there exists a fully developed theory [25], [32], and Theorem 1 gives a complete characterization of strong solvability. However, if $\mathcal{I}=\mathbb{N}_0$, the set of exponentials $\{\mathrm{e}^{ikt}\}_{k\in\mathbb{N}_0}$ is no longer complete. This make a huge difference from a mathematical point of view. As a consequence, the theory and proof techniques from [25], [32] can no longer be used, and new approaches are needed.

In the rest of this paper we will study the CF problem for OFDM and analyze the effects of restricting the carriers to the positive frequencies, i.e., choosing $\mathcal{I}=\mathbb{N}_0$. In order to compare the two scenarios, $\mathcal{I}=\mathbb{Z}$ (full carrier set) and $\mathcal{I}=\mathbb{N}_0$ (restricted carrier set), we will always assume that the information set $\mathcal{K}$ is a subset of $\mathbb{N}_0$. As compensation sets we consider the full set $\mathcal{K}_{\mathbb{Z}}^{\complement}=\mathbb{Z}\backslash\mathcal{K}$ if $\mathcal{I}=\mathbb{Z}$, and the restricted set $\mathcal{K}_{\mathbb{N}_0}^{\complement}=\mathbb{N}_0\setminus\mathcal{K}$ if $\mathcal{I}=\mathbb{N}_0$.

Definition 3 (Strong Solvability of the OFDM CF problem with restricted carrier set)**.** For a set $\mathcal{K}\subset\mathbb{N}_0$, we say that the OFDM CF problem with restricted carrier set is strongly solvable with finite extension constant $C_{\mathrm{Ex}}^{\mathbb{N}_0}$, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}_{\mathbb{N}_0}^{\complement})$, such that

$$\left\|\sum_{k\in\mathcal{K}} a_k\,\mathrm{e}^{ik\,\cdot}+\sum_{k\in\mathcal{K}_{\mathbb{N}_0}^{\complement}} b_k\,\mathrm{e}^{ik\,\cdot}\right\|_{L^\infty}\le C_{\mathrm{Ex}}^{\mathbb{N}_0}\|a\|_{\ell^2(\mathcal{K})}. \quad (7)$$

We call the OFDM CF problem with restricted carrier set strongly solvable if it is strongly solvable for some finite extension constant $C_{\mathrm{Ex}}^{\mathbb{N}_0}$.

Remark 3. Clearly, for a given information set $\mathcal{K}\subset\mathbb{N}_0$, a necessary condition for the strong solvability of the OFDM CF problem with restricted carrier set, is the strong solvability of the OFDM CF problem with full carrier set, because $\mathcal{K}_{\mathbb{N}_0}^{\complement}=\mathbb{N}_0\setminus\mathcal{K}\subset\mathbb{Z}\setminus\mathcal{K}=\mathcal{K}_{\mathbb{Z}}^{\complement}$.

We are interested in a characterization similar to Theorem 1 for the strong solvability of the CF problem with restricted carrier set. The subspace

$$\mathfrak{F}^1(\mathcal{K})=\left\{f\in L^1[-\pi,\pi]\colon\right.$$
$$\left. f=\sum_{k\in\mathcal{K}} a_k\,\mathrm{e}^{ik\,\cdot}\quad\text{for some }\{a_k\}_{k\in\mathcal{K}}\subset\mathbb{C}\right\},$$

which is defined similar to (4), will play an important role in the analysis. Our main result is the following theorem, a characterization of the strong solvability of the CF problem with restricted carrier set.

Theorem 2. *Let $\mathcal{K}\subset\mathbb{N}_0$. The OFDM CF problem with restricted carrier set ($\mathcal{I}=\mathbb{N}_0$) is strongly solvable if and only if there exists a constant C_1 such that*

$$\|f\|_{L^2}\le C_1\|f\|_{L^1} \quad (8)$$

for all $f\in\mathfrak{F}^1(\mathcal{K})$.

The proof of Theorem 2 will be given in Appendix D.

Using the result from [20], [26], i.e., Theorem 1, we immediately obtain the following corollary.

Corollary 1. *Let $\mathcal{K}\subset\mathbb{N}_0$. The OFDM CF problem with restricted carrier set ($\mathcal{I}=\mathbb{N}_0$) is strongly solvable if and only if it is strongly solvable with full carrier set ($\mathcal{I}=\mathbb{Z}$).*

Corollary 1 shows that with respect to strong solvability, the usage of the full compensation set $\mathcal{K}_{\mathbb{Z}}^{\complement}$ does not give any advantage over the usage of the restricted compensation set $\mathcal{K}_{\mathbb{N}_0}^{\complement}$. This is surprising, because intuitively one would assume that using the full compensation set gives a higher degree of flexibility in the design of the compensation signal and hence a better compensation. Nevertheless, we conjecture that in general better constants can be achieved when using the full compensation set $\mathcal{K}_{\mathbb{Z}}^{\complement}$.

Proof of Corollary 1. "⇒": This direction is obvious. "⇐": If the OFDM CF problem is strongly solvable with full carrier set then we have (5) for all $f\in\mathfrak{F}^1(\mathcal{K})$, according to Theorem 1. Application of Theorem 2 completes the proof. □

V. Discussion: Solvability With Restricted Compensation Set

In this section we discuss why we cannot infer the solvability of the CF problem with restricted carrier set from the solvability of the CF problem with full carrier set.

Let $\mathcal{G}$ be an arbitrary subset of $\mathbb{Z}$, and define

$$(P_{\mathcal{G}}f)(\theta)=\sum_{k\in\mathcal{G}} c_k(f)\,\mathrm{e}^{ik\theta},\quad \theta\in[-\pi,\pi], \quad (9)$$

where

$$c_k(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta.$$

First we consider $P_{\mathcal{G}}$ as an operator mapping from $L^2[-\pi,\pi]$ into $L^2[-\pi,\pi]$. Using Parseval's equality, we see that

$$\|P_{\mathcal{G}}f\|_{L^2}^2=\sum_{k\in\mathcal{G}}|c_k(f)|^2\le\sum_{k\in\mathbb{Z}}|c_k(f)|^2=\|f\|_{L^2}^2.$$

Clearly, $P_{\mathcal{G}}\colon L^2[-\pi,\pi]\to L^2[-\pi,\pi]$ is a projection operator, i.e., a linear and bounded operator that satisfies $P_{\mathcal{G}}(P_{\mathcal{G}}f)=P_{\mathcal{G}}f$ for all $f\in L^2[-\pi,\pi]$. The series in (9) converges in the L^2 norm. However, for us, instead of $L^2[-\pi,\pi]$, the relevant space is $L^\infty[-\pi,\pi]$, and $P_{\mathcal{G}}$ does not necessarily map $L^\infty[-\pi,\pi]$ into $L^\infty[-\pi,\pi]$. Hence, if we solve the CF problem with full carrier set, we cannot simply project the solution onto the positive frequencies, because for $f\in L^\infty[-\pi,\pi]$, $P_{\mathbb{N}_0}f$ is not necessarily in $L^\infty[-\pi,\pi]$.

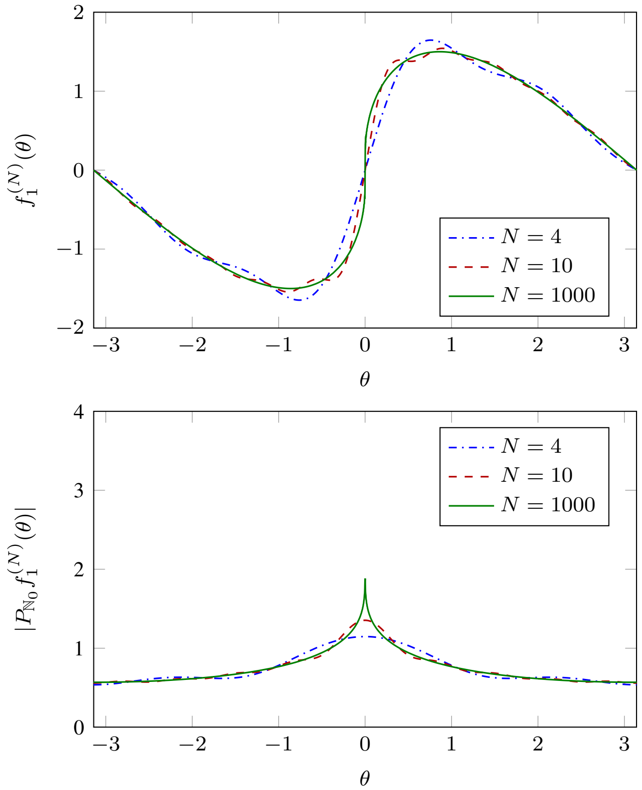

Fig. 1. Plot of the signal $f_1^{(N)}(\theta) = \sum_{k=1}^{N} \sin(k\theta)/(k\log(1+k))$ (upper panel) and the absolute value of its projection $|(P_{\mathbb{N}_0} f_1^{(N)})(\theta)|$ (lower panel) for $N = 4, 10, 1000$.

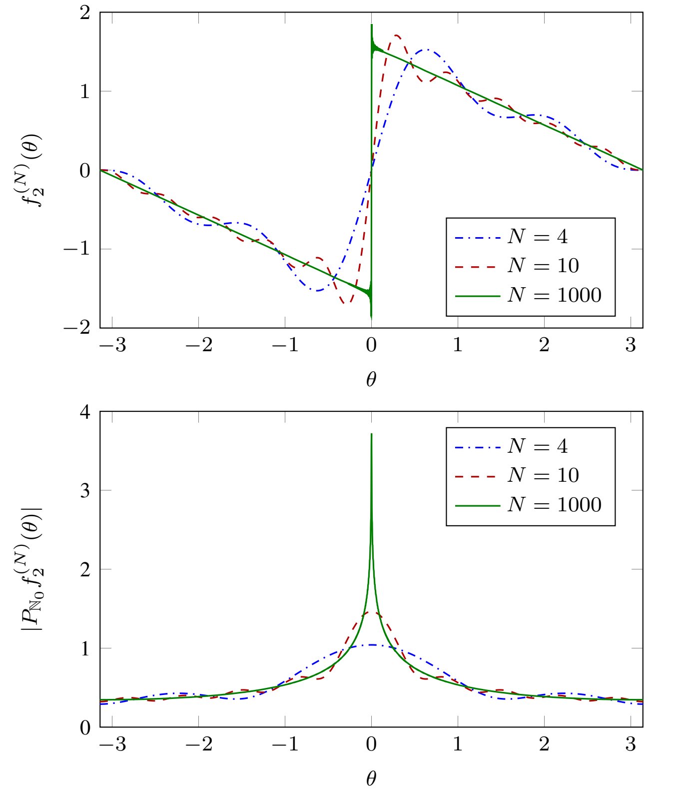

Fig. 2. Plot of the signal $f_2^{(N)}(\theta) = \sum_{k=1}^{N} \sin(k\theta)/k$ (upper panel) and the absolute value of its projection $|(P_{\mathbb{N}_0} f_2^{(N)})(\theta)|$ (lower panel) for $N = 4, 10, 1000$.

We illustrate this behavior with two examples. Let

$$f_1(\theta) = \sum_{k=1}^{\infty} \frac{1}{k\log(1+k)} \sin(k\theta), \quad \theta \in [-\pi, \pi],$$

and

$$f_2(\theta) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\theta), \quad \theta \in [-\pi, \pi].$$

Note that f_1 is absolutely continuous and f_2 is bounded by a constant. Then we have

$$(P_{\mathbb{N}_0} f_1)(\theta) = \sum_{k=1}^{\infty} \frac{1}{2ik\log(1+k)} \mathrm{e}^{ik\theta}, \quad \theta \in [-\pi, \pi],$$

and

$$(P_{\mathbb{N}_0} f_2)(\theta) = \sum_{k=1}^{\infty} \frac{1}{2ik} \mathrm{e}^{ik\theta}, \quad \theta \in [-\pi, \pi].$$

Both series converge in $L^2[-\pi,\pi]$, however, we have $P_{\mathbb{N}_0} f_1 \notin L^\infty[-\pi,\pi]$ and $P_{\mathbb{N}_0} f_2 \notin L^\infty[-\pi,\pi]$, because both functions are unbounded in the vicinity of zero. This behavior is illustrated in Figures 1 and 2, where partial sums of the series and absolute values of their projections are plotted.

VI. Applications and Examples

For practical applications it is important to have guidelines how to choose the information set $\mathcal{K}$. In this section we present several examples for the choice of the information set and discuss the solvability of the OFDM CF problem for these choices.

A. Two Negative Examples

First, we present two examples that demonstrate how not to choose the set $\mathcal{K}$. While this gives no immediate indication of a good set $\mathcal{K}$, it at least significantly reduces the set of information sets that have to be considered in the search for good information sets.

Our approach enables us to connect the OFDM CF problem with the theory of arithmetic progressions. This theory is a very active area of mathematics with several deep results [35]–[38]. For example, Szemerédi's theorem on arithmetic progressions is one of the key results in combinatorics [39]. We can use the results on arithmetic progressions in order to obtain further insights into the OFDM CF problem.

Definition 4. An arithmetic progression of length $L \in \mathbb{N}$ is a subset of $\mathbb{Z}$, having the form

$$\{a, a+d, a+2d, \dots, a+(L-1)d\}$$

for some $a \in \mathbb{Z}$ and $d \in \mathbb{N}$.

The length of the largest arithmetic progression in the information set $\mathcal{K}$ influences the size of the optimal extension constant. For the OFDM CF reduction problem with full compensation set ($\mathcal{I} = \mathbb{Z}$), a lower bound for the smallest extension constant is given by

$$\underline{C}_{\text{Ex}}^{\mathbb{Z}} > \frac{\sqrt{L}}{\frac{4}{\pi}\log\left(\frac{L}{2}\right) + 4 + \frac{2}{24-\pi^2} + \frac{4}{\pi^2}}, \tag{10}$$

when $\mathcal{K}$ contains an arithmetic progression of length L [40]. Note that according to the definition of the smallest extension constant $\underline{C}_{\text{Ex}}^{\mathbb{Z}}$, this implies that there exists an information sequence $a \in \ell^2(\mathcal{K})$ with $\|a\|_{\ell^2(\mathcal{K})} = 1$ such that

$$\left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}_{\mathbb{Z}}^{\complement}} b_k\phi_k \right\|_{L^\infty} \geq \frac{\sqrt{L}}{\frac{4}{\pi}\log\left(\frac{L}{2}\right) + 4 + \frac{2}{24-\pi^2} + \frac{4}{\pi^2}}$$

for all compensation sequences $b \in \ell^2(\mathcal{K}_{\mathbb{Z}}^{\complement})$. This shows that long arithmetic progressions should be avoided in the information set $\mathcal{K}$, because they can lead to large peak values. The previous discussion was for the full carrier set ($\mathcal{I} = \mathbb{Z}$), but clearly the situation cannot improve if the restricted carrier set ($\mathcal{I} = \mathbb{N}$) is considered.

We have the following interesting result by Terence Tao about the existence of arithmetic progressions [36, Theorem 1.2, p. 2].

Theorem 3 (Quantitative form of Szemerédi's theorem)**.** *Let $0 < \delta < 1$ and $L \in \mathbb{N}$. There exists a natural number $N_0 = N_0(\delta, L)$ such that for all $N \geq N_0$ we have: If $\mathcal{K} \subset \{0,\dots,N\}$ satisfies $|\mathcal{K}| \geq \delta N$ then $\mathcal{K}$ contains an arithmetic progression of length L.*

This theorem shows that if we choose the set $\mathcal{K} \subset \{0,\dots,N\}$ not too thin and if N is large enough than $\mathcal{K}$ contains an arithmetic progression of length L. For practical applications, it is desirable to chose δ large.

Now we will use the lower bound on the optimal extension constant (10) together with Theorem 3 to obtain a necessary condition for the strong solvability [25], [26].

Theorem 4. *Let $\mathcal{K} \subset \mathbb{N}_0$. If the OFDM CF problem is strongly solvable then we have*

$$\lim_{N\to\infty} \frac{|\mathcal{K} \cap \{0,\dots,N\}|}{N+1} = 0. \tag{11}$$

Hence, the density of the information set $\mathcal{K}$ has to go to zero in order that the OFDM CF problem is solvable. In other words, if (11) is not satisfied, i.e., if we have

$$\limsup_{N\to\infty} \frac{|\mathcal{K} \cap \{0,\dots,N\}|}{N+1} > 0,$$

then $\mathcal{K}$ contains arbitrarily long arithmetic progressions and the condition for strong solvability cannot be fulfilled [25]. From this observation, we immediately obtain the following example.

Example 1. Let $\mathcal{K}$ be the set of all even natural numbers. Then the OFDM CF problem is not strongly solvable.

Note that condition (11) is not sufficient for solvability. There exist sets $\mathcal{K}$ that satisfy (11), but for which the OFDM CF problem is not strongly solvable. For example, if $\mathcal{K}$ is the set of primes $\mathcal{P}$, then we have

$$\limsup_{N\to\infty} \frac{|\mathcal{P} \cap \{0,\dots,N\}|}{N+1} = 0.$$

However, a result by Green and Tao [41] shows that the set of primes $\mathcal{P}$ contains arbitrarily long arithmetic progressions. Hence, by the same arguments that were used to derive Theorem 4, it follows that the OFDM CF problem is not solvable.

Example 2. Let $\mathcal{K}$ be the set of all primes. Then the OFDM CF problem is not strongly solvable.

B. A Probabilistic Approach

Another approach to select the information set, is random selection. In this approach, we first choose the expected density η that our information set $\mathcal{K}$ should have. In other words η is the expectation of the fraction of information carriers to the total number of carriers. Then the information carriers are selected randomly according to the following procedure. Assume that we have $N+1$ carriers $\{0,\dots,N\}$ available in total. With a probability $p = \eta$, we use any of those carriers as an information carrier, i.e., include it in the set $\mathcal{K}$. Then, we have $\eta = \mathbb{E}(|\mathcal{K}|/(N+1))$. The compensation set is given by $\mathcal{K}_{\{0,\dots,N\}}^{\complement} = \{0,\dots,N\} \setminus \mathcal{K}$. It is clear that the selection probability should be chosen depending on N in order to get the strongest possible result.

Using this randomized procedure, one could hope that in average the performance is good. However, this is not the case because of the following theorem, which was derived in [42] for the full carrier set ($\mathcal{I} = \mathbb{Z}$), based on a result by Conlon and Gowers [43] and Schacht [44].

Theorem 5. *Let $L \in \mathbb{N}$. There exists a constant C such that for every sequence $\{p_N\}_{N\in\mathbb{N}}$ with $p_N \geq C/N^{\frac{1}{L-1}}$, $N \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \mathbb{P}(A_{N,L,p_N}) = 1,$$

where A_{N,L,p_N} denotes the event: "The OFDM CF problem is not solvable with an extension constant

$$C_{Ex}^{\{0,\dots,N\}} \leq \frac{\sqrt{L}}{\frac{4}{\pi}\log\left(\frac{L}{2}\right) + 4 + \frac{2}{24-\pi^2} + \frac{4}{\pi^2}} \tag{12}$$

for the information set $\mathcal{K} \subset \{0,\dots,N\}$, chosen as described above using the probability p_N.

The theorem shows that a probabilistic choice of the information set leads, with a probability close to one, to a performance that is as bad as the performance in the case where the information set contains an arithmetic progression.

Remark 4. Theorem 5 is a direct consequence of [42], because for the restricted carrier set ($\mathcal{I} = \mathbb{N}_0$) the smallest extension constant is always larger than or equal to the smallest extension constant for the full carrier set ($\mathcal{I} = \mathbb{Z}$).

Remark 5. We directly use the theorems by Conlon and Gowers [43] and Schacht [44], respectively. Let B_{N,L,p_N} denote the event that "the set $\mathcal{K}$ contains an arithmetic progression of length L". Then if $p_N \geq C/N^{\frac{1}{L-1}}$ we have

$$\lim_{N\to\infty} \mathbb{P}(B_{N,L,p_N}) = 1.$$

Hence, the extension constant $C_{\text{Ex}}^{\{0,\dots,N\}}$ must be larger that the right-hand side of (12). If $p_N \geq C/N^{\frac{1}{L-1}}$ then we have

$$\mathbb{E}(|\mathcal{K}|) \geq CN^{1-\frac{1}{L-1}} = CN^{\frac{L-2}{L-1}}.$$

That is, compared to Szemerédi's theorem, we have a statement for a density $\delta_{N,L} = C/N^{\frac{1}{L-1}}$. However, the statement is weaker: For fixed L the statement is only for almost all subsets $\mathcal{K} \subset \{0, \ldots, N\}$, where the probability measure has to satisfy $p_N \geq C/N^{\frac{1}{L-1}}$.

Remark 6. In information theory one often faces the situation that optimal codes with a closed form description can only be derived for relatively few operational tasks. However, if the "correct" counting measure is introduced on the combinatoric objects, then almost all combinatoric objects have the desired property, i.e., are capacity achieving codes for example. Similar statements are true for example for spreading codes for CDMA systems with good correlation properties. For random sets $\mathcal{K} \subset \{0, \ldots, N\}$, this is no longer true if the sets are chosen independent and identically distributed with $p \geq C/N^{\frac{1}{L-1}}$. Practically, this is no restriction if $\mathbb{E}(|\mathcal{K}|)$ shall not be too small. $\mathbb{E}(|\mathcal{K}|)$ should be comparable to N, but in this case random constructions are useless for the CF reduction.

C. A Positive Example

The condition (8) in Theorem 2 completely specifies the information sets $\mathcal{K}$ for which the OFDM CF problem is strongly solvable. In the following, we present an information set for which the OFDM CF problem is strongly solvable.

Example 3. Let $\mathcal{K} = \{2^k\}_{k \in \mathbb{N}_0}$. Then the OFDM CF problem is strongly solvable.

Since $\{2^k\}_{k \in \mathbb{N}_0}$ is a lacunary sequence, there exists a constant C_1 such that we have (8) for all $f \in \mathfrak{F}^1(\mathcal{K})$. For details, please see [45, p. 240].

Although the OFDM CF problem is strongly solvable for $\mathcal{K} = \{2^k\}_{k \in \mathbb{N}_0}$, it is not advisable to use this information set in practice. The density of this set gets smaller and smaller very quickly, because the distance of subsequent carriers grows exponentially.

VII. Theory for Weak Solvability

In this section we analyze the weak solvability of the OFDM CF problem.

Definition 5 (Weak solvability of the OFDM CF problem with restricted carrier set)**.** For a set $\mathcal{K} \subset \mathbb{N}_0$, we say that the CF problem with restricted carrier set is weakly solvable if for all $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}^{\complement}_{\mathbb{N}_0})$ such that

$$\left\| \sum_{k \in \mathcal{K}} a_k \, \mathrm{e}^{ik\,\cdot} + \sum_{k \in \mathcal{K}^{\complement}_{\mathbb{N}_0}} b_k \, \mathrm{e}^{ik\,\cdot} \right\|_{L^\infty} < \infty.$$

Weak solvability is indeed a weaker form of solvability, because strong solvability always implies weak solvability. In [32] weak solvability of the CF problem was analyzed for OFDM, i.e., the system of exponential functions $\{\mathrm{e}^{ikt}\}_{k \in \mathbb{Z}}$, and the full carrier set ($\mathcal{I} = \mathbb{Z}$). As a surprising result it turned out that in this setting weak solvability implies strong solvability, i.e., that both solvability concepts are equivalent. In [46] this result was generalized to arbitrary compete orthonormal systems. Since in our setting with restricted carrier set ($\mathcal{I} = \mathbb{N}_0$), the system of exponential functions is no longer complete, we cannot directly use the results from [32], [46].

Using a modified version of the proof in [46], we can derive the interesting result that weak solvability implies strong solvability also for the OFDM CF problem with restricted carrier set. Hence, also in our setting both concept are equivalent.

Theorem 6. *Let $\mathcal{K} \subset \mathbb{N}_0$. If the OFDM CF problem with restricted carrier set ($\mathcal{I} = \mathbb{N}_0$) is weakly solvable then it is also strongly solvable.*

The proof of Theorem 6 is given in Appendix E.

Remark 7. We conjecture that the result of Theorem 6, i.e., the equivalence of strong and weak solvability, does not hold for arbitrary carrier sets $\mathcal{I} \subset \mathbb{Z}$. In our proof, which employs the theory of Hardy spaces, it is essential to have the set of exponentials and $\mathcal{I} = \mathbb{N}_0$.

In the remainder of this section we analyze the size of the set of information sequences, for which the CF problem with restricted carrier set is not solvable. For a given set $\mathcal{K} \subset \mathcal{I}$, let

$$\mathcal{B}_{\mathcal{I}}(a) = \left\{ b \in \ell^2(\mathcal{K}^{\complement}_{\mathcal{I}}) \colon \left\| \sum_{k \in \mathcal{K}} a_k \, \mathrm{e}^{ik\,\cdot} + \sum_{k \in \mathcal{K}^{\complement}_{\mathcal{I}}} b_k \, \mathrm{e}^{ik\,\cdot} \right\|_{L^\infty} < \infty \right\}.$$

We have $\mathcal{B}_{\mathcal{I}}(a) = \emptyset$ if and only if the CF problem with compensation set $\mathcal{K}^{\complement}_{\mathcal{I}}$ is not weakly solvable. By

$$\mathcal{D}_{\mathcal{I}} = \{ a \in \ell^2(\mathcal{K}) \colon \mathcal{B}_{\mathcal{I}}(a) = \emptyset \}$$

we denote the set of information sequences, for which the CF problem with compensation set $\mathcal{K}^{\complement}_{\mathcal{I}}$ is not solvable.

In [32] the set $\mathcal{D}_{\mathbb{Z}}$ was analyzed. It was shown for the full carrier set ($\mathcal{I} = \mathbb{Z}$) that if the OFDM CF problem is not weakly solvable, then the set $\mathcal{D}_{\mathbb{Z}}$ is a residual set, i.e., is big in a topological sense.

The next theorem shows that the same results holds for the OFDM CF problem with restricted carrier set.

Theorem 7. *Let $\mathcal{K} \subset \mathbb{N}_0$. If the OFDM CF problem with restricted carrier set ($\mathcal{I} = \mathbb{N}_0$) is not weakly solvable then $\mathcal{D}_{\mathbb{N}_0}$ is a residual set in $\ell^2(\mathcal{K})$.*

The proof of Theorem 7 is given in Appendix F.

VIII. Conclusion

While almost all existing publications treat tone reservation for OFDM in a numerical manner, we provided an analytical approach in this paper. For the ODFM CF problem with restricted carrier set we studied and answered the fundamental question "When is the CF problem solvable?". This is a first step towards solving the three general questions presented in the introduction.

For our analysis we specifically used the properties of the exponential functions that are used in OFDM and the choice $\mathcal{I} = \mathbb{N}_0$ of the carrier set. We derived a geometric functional analytic description for the OFDM CF problem with restricted carrier set. We also provided several examples that show how our theory can be applied to derive guidelines for the choice of optimal information sets in the practical finite setting. To the best of our knowledge this is the fist analytical result into this direction.

It is an open questions whether results, similar to those obtained in this paper, are true for other carrier sets and other ONSs. In particular a theory for arbitrary carrier sets, i.e., for arbitrary $\mathcal{I} \subset \mathbb{Z}$, would be useful. Our result, i.e., the restriction to $\mathcal{I} = \mathbb{N}_0$ is a first step into this direction. We hope that our results and proof techniques can also serve as a starting point for further research.

Appendix A
Basics on Hardy Spaces

In this section we introduce the necessary notions and results from complex analysis. We roughly follow the presentation and notation in [47]. We introduce the Hardy spaces next. For $1 \leq p < \infty$, $H^p(\mathbb{D})$ is the space of all functions f analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ for which

$$\|f\|_{H^p(\mathbb{D})} = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r\,\mathrm{e}^{i\theta})|^p \,\mathrm{d}\theta \right)^{\frac{1}{p}} < \infty.$$

$H^\infty(\mathbb{D})$ is the space of all bounded analytic functions on $\mathbb{D}$, equipped with the norm $\|f\|_{H^\infty(\mathbb{D})} = \sup_{|z|<1} |f(z)|$. For $f \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the radial limit $\lim_{r\to 1} f(r\,\mathrm{e}^{i\theta})$ exists for almost all $\theta \in \mathbb{T}$, and we denote it by $f(\mathrm{e}^{i\theta})$. We have $f(\mathrm{e}^{i\,\cdot}) \in L^p(\mathbb{T})$, as well as $\|f(\mathrm{e}^{i\,\cdot})\|_{L^p} = \|f\|_{H^p(\mathbb{D})}$. Further, it holds for all $f \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$, that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\mathrm{e}^{i\theta})\,\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta} - z} \,\mathrm{d}\theta, \quad |z| < 1. \tag{13}$$

Hence, we can identify $f \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$, with its boundary function $f(\mathrm{e}^{i\theta})$. By $H^p(\mathbb{T})$ we denote the set of boundary functions $f(\mathrm{e}^{i\,\cdot})$ of functions $f \in H^p(\mathbb{D})$.

For $1 \leq p \leq \infty$, $L^p(\mathbb{T})$ functions are uniquely determined by their Fourier coefficients

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-ik\theta} \,\mathrm{d}\theta, \quad k \in \mathbb{Z}, \tag{14}$$

and, for $1 < p < \infty$, we have

$$f(\theta) \overset{L^p}{=} \sum_{k=-\infty}^{\infty} c_k(f)\,\mathrm{e}^{ik\theta}, \quad \theta \in [-\pi, \pi].$$

For $1 \leq p \leq \infty$ we have the following characterization of Hardy spaces

$$H^p(\mathbb{T}) = \left\{ f \in L^p(\mathbb{T})\colon \int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-ik\theta} \,\mathrm{d}\theta = 0, \quad k < 0 \right\}.$$

Thus, $H^p(\mathbb{T})$, $1 \leq p \leq \infty$, is a closed subspace of $L^p(\mathbb{T})$. We use the abbreviations $H^p := H^p(\mathbb{T})$ and $L^p := L^p(\mathbb{T})$. Let

$$H_0^1 = \left\{ f \in H^1\colon \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \,\mathrm{d}\theta = 0 \right\}$$

be the set of all functions in H^1 whose zeroth Fourier coefficient is zero. H_0^1 is a closed subspace of H^1, and therefore H_0^1 is a closed subspace of L^1.

Further, we need the quotient space L^1/H_0^1, consisting of the set of all equivalence classes

$$[f] = f + H_0^1 = \{f + h\}_{h \in H_0^1}, \quad f \in L^1.$$

Note that $[f]$ is the set of all L^1-functions q with $c_k(q) = c_k(f)$ for all $k \leq 0$. Since L^1 is complete and H_0^1 is a closed subspace of L^1, it follows that L^1/H_0^1 is a Banach space, when equipped with the norm

$$\|[f]\|_{L^1/H_0^1} = \inf_{h \in H_0^1} \|f + h\|_{L^1}.$$

A function $f\colon \partial\mathbb{D} \to \mathbb{C}$, where $\partial\mathbb{D} = \{z \in \mathbb{C}\colon |z| = 1\}$ denotes unit circle, is said to belong to the space of functions of bounded mean oscillation BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|} \int_I |f(z) - m_I(f)| \,\mathrm{d}z < \infty,$$

where I denotes an arc on $\partial\mathbb{D}$, $|I|$ is the length of I, and $m_I(f) = 1/|I| \int_I f(z) \,\mathrm{d}z$.

By

$$(P_+ f)(z) = \sum_{k=0}^{\infty} c_k(f(\mathrm{e}^{i\,\cdot}))z^k, \quad |z| < 1,$$

and

$$(P_- f)(z) = \sum_{k=-\infty}^{-1} c_k(f(\mathrm{e}^{i\,\cdot}))z^k, \quad |z| > 1,$$

we denote the Riesz projections. The boundary values $(P_+ f)(\mathrm{e}^{i\theta})$ exist almost everywhere and uniquely determine $P_+ f$. For $1 < p < \infty$, $P_+\colon L^p(\partial\mathbb{D}) \to H^p(\mathbb{D})$ is a bounded linear operator. For $p = 1$ and $p = \infty$ this is no longer the case, as we have already observed in Section V. For $p = 1$ we have the weak-type estimate [47, p. 106]: There exists a constant $C_2 > 0$ such that

$$\mu(\{\theta \in [-\pi, \pi]\colon |(P_+ f)(\mathrm{e}^{i\theta}) > \lambda|\}) \leq \frac{C_2}{\lambda} \|f\|_{L^1}. \tag{15}$$

For $p = \infty$ we have

$$\|P_+ f\|_{\mathrm{BMO}} \leq C_3 \|f\|_{L^\infty}.$$

For $1 \leq p \leq \infty$, $P_+ f$ has a simple Cauchy integral representation

$$(P_+ f)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\mathrm{e}^{i\theta})\,\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta} - z} \,\mathrm{d}\theta, \quad |z| < 1.$$

For $p = 2$ the projection P_+ is particularly simple, however, we need to understand the case $p = 1$. Note that P_+ does not map $L^1(\partial\mathbb{D})$ into $H^1(\mathbb{D})$. The Riesz projection P_- has properties that are analogous to those of P_+. In particular, there exists a constant $C_4 > 0$ such that

$$\mu(\{\theta \in [-\pi, \pi]\colon |(P_- f)(\mathrm{e}^{i\theta}) > \lambda|\}) \leq \frac{C_4}{\lambda} \|f\|_{L^1}. \tag{16}$$

Since H_0^1 is a closed subspace of L^1, we can define the metric projection according to Kahane [48]. For $f \in L^1$, let

$$d(f, H_0^1) = \inf_{h \in H_0^1} \|f + h\|_{L^1}$$

denote the smallest distance between f and H_0^1. The mapping

$$P(f) = \{h \in H_0^1\colon \|f - h\|_{L^1} = d(f, H_0^1)\}$$

is called metric projection of f on H_0^1. In our case, it can be shown that, for each $f \in L^1$, $P(f)$ contains exactly one element [48]. Further, P is a continuous operator. However, P is a non-linear operator in general.

The previous discussion shows that for each $f \in L^1$ there exists exactly one $h_* \in H_0^1$, such that

$$\|[f]\|_{L^1/H_0^1} = \|f + h_*\|_{L^\infty}.$$

$h_* = P(f)$ is the metric projection of f on H_0^1, and h_* depends continuously on f. Clearly, we have $f + h_* \in [f]$.

In the remainder of this section we will discuss the connections between the Riesz projection P_+ and the Hilbert transform, which is of crucial importance in the information technology field. For $1 < p < \infty$, the Hilbert transform is defined by

$$(Hf)(\mathrm{e}^{i\tau}) = \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{f(\mathrm{e}^{i\theta})}{\tan\left(\frac{\tau-\theta}{2}\right)} \,\mathrm{d}\theta,$$

where the singular integral, i.e., the Cauchy principal value exists almost everywhere. For $1 < p < \infty$, the Hilbert transform $H \colon L^p(\partial\mathbb{D}) \to L^p(\partial\mathbb{D})$ is a bounded linear operator.

For all $f \in L^p(\partial\mathbb{D})$, the expression (13) defines a bounded linear operator $Q \colon L^p(\partial\mathbb{D}) \to H^p(\mathbb{D})$ with

$$Qf = c_0(f) + \frac{1}{2}(Id - iH)f.$$

We further have

$$(P_+f)(z) = \sum_{n=0}^{\infty} c_n(f) z^n, \quad |z| < 1.$$

Using

$$(H\,\mathrm{e}^{ik\,\cdot})(\theta) = -i\,\mathrm{sgn}(k)\,\mathrm{e}^{ik\theta},$$

where sgn denotes the signum function with $\mathrm{sgn}(0) = 0$, we have for

$$g_k(\mathrm{e}^{i\theta}) = \mathrm{e}^{ik\theta}$$

that

$$(Qg_k)(z) = c_0(g_k) + \frac{1}{2}\left(1 - i^2\,\mathrm{sgn}(k)\right) g_k(z).$$

It follows that $(Qg_k)(z) = c_0(g_k)$ if $k = 0$. For $k > 0$ we have $c_0(g_k) = 0$ and $\frac{1}{2}\left(1 - i^2\,\mathrm{sgn}(k)\right) = 1$, which shows that $(Qg_k)(z) = g_k(z)$. For $k < -1$ we have $(Qg_k)(z) = 0$, $|z| < 1$, because $1 + \mathrm{sgn}(k) = 0$. Thus, we have

$$Qg_k = P_+g_k$$

for all $k \in \mathbb{Z}$. Since the set of trigonometric polynomials is dense in $L^p(\partial\mathbb{D})$, $1 \le p < \infty$, we have

$$Qf = P_+f \tag{17}$$

for all $f \in L^p(\partial\mathbb{D})$, $1 \le p < \infty$. Moreover, since $L^\infty(\partial\mathbb{D}) \subset L^p(\partial\mathbb{D})$, $p < \infty$, the equality (17) also holds for all $f \in L^\infty(\partial\mathbb{D})$.

Furthermore, the Hilbert transform is no bounded operator from $L^1(\partial\mathbb{D})$ to $L^1(\partial\mathbb{D})$, $L^\infty(\partial\mathbb{D})$ to $L^\infty(\partial\mathbb{D})$, and $C(\partial\mathbb{D})$ to $C(\partial\mathbb{D})$. This is the problem we encountered and that was visualized in Figures 1 and 2.

Appendix B
A Naive Approach

Before we prove Theorem 2, we want to discuss a naive approach. Although this naive approach does not work, it will provide us valuable insights. In particular, we will see the problems that appear if the path that was used in [25] is taken.

We only discuss the "⇐" direction of the proof here, because the "⇒" direction is easy, as we will see in Appendix D. Let

$$\overline{\mathfrak{F}}^1(\mathcal{K}) = \left\{f \in L^1 \colon c_k(f) = 0 \text{ for } k \in \mathbb{Z} \setminus (-\mathcal{K})\right\}, \tag{18}$$

where

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-ik\theta} \,\mathrm{d}\theta.$$

That is, $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$ is concentrated on $-\mathcal{K}$. Clearly, we have (8) for all $f \in \mathfrak{F}^1(\mathcal{K})$ if and only if we have (8) for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Hence, in order to prove the "⇐" direction, we can assume that (8) holds for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$.

For $a = \{a_k\}_{k \in \mathcal{K}} \in \ell^2(\mathcal{K})$ and $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$ we define the linear functional

$$\Phi_a(f) = \sum_{k \in \mathcal{K}} c_{-k}(f) a_k.$$

Since

$$\begin{aligned} |\Phi_a(f)| &\le \left(\sum_{k\in\mathcal{K}} |c_{-k}(f)|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathcal{K}} |a_k|^2\right)^{\frac{1}{2}} \\ &= \|f\|_{L^2}\|a\|_{\ell^2(\mathcal{K})} \\ &\le C_1\|f\|_{L^1}\|a\|_{\ell^2(\mathcal{K})}, \end{aligned}$$

we see that the linear functional $\Phi_a \colon \overline{\mathfrak{F}}^1(\mathcal{K}) \to \mathbb{C}$ is well defined and bounded. $\overline{\mathfrak{F}}^1(\mathcal{K})$ is a closed subspace of L^1. Hence, according to the Hahn–Banach theorem, we can extend the functional Φ_a to a functional Φ_a^{Ex} on all of L^1 such that the norm is retained. That is, we have

$$|\Phi_a^{\mathrm{Ex}}(f)| \le C_1\|f\|_{L^1}\|a\|_{\ell^2(\mathcal{K})}$$

for all $f \in L^1$. The Riesz representation theorem implies that there exists a function $g \in L^\infty$ with $\|g\|_{L^\infty} \le C_1\|a\|_{\ell^2(\mathcal{K})}$, such that

$$\Phi_a^{\mathrm{Ex}}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) g(\theta) \,\mathrm{d}\theta$$

for all $f \in L^1$. For $k \in \mathcal{K}$ and $f_k(\theta) = \mathrm{e}^{-ik\theta}$, $\theta \in [-\pi, \pi]$, we have $f_k \in \overline{\mathfrak{F}}^1(\mathcal{K})$, and it holds that

$$\Phi_a^{\mathrm{Ex}}(f_k) = \Phi_a(f_k) = \sum_{l\in\mathcal{K}} a_l \frac{1}{2\pi} \int_{-\pi}^{\pi} f_k(\theta)\,\mathrm{e}^{il\theta} \,\mathrm{d}\theta = a_k$$

for all $k \in \mathcal{K}$. Thus, $c_k(g) = a_k$ and $g \in L^\infty$, i.e., g solves the CF problem. However, the spectrum of g is not necessarily concentrated on $\mathbb{N}_0$, but rather on $\mathbb{Z}$, i.e., with this approach we use $\mathcal{K}_{\mathbb{Z}}^{\complement} = \mathbb{Z} \setminus \mathcal{K}$ as compensation set in general.

A first idea that might come into one's mind is to use the signal

$$g_+(\mathrm{e}^{i\theta}) \overset{L^2}{=} \sum_{k=0}^{\infty} c_k(g)\,\mathrm{e}^{ik\theta}.$$

However, g_+ is only in BMO and not in $H^\infty(\mathbb{D})$ in general. Although we control the BMO-norm according to $\|g_+\|_{\mathrm{BMO}} \leq C_1\|a\|_{\ell^2(\mathcal{K})}$, we have no control of the L^∞-norm because the BMO norm controls the L^p-norms only for $p < \infty$ and not for $p = \infty$. Thus, this approach does not work.

In order to prove Theorem 2, we need to go a different path which requires the introduction of suitable function spaces. We use the quotient space L^1/H^1_0. This is expedient because the dual space of L^1/H^1_0, which we denote by $(L^1/H^1_0)^*$ is H^∞. However, compared to the scenario where we use the full compensation set $\mathcal{K}^{\complement}_{\mathbb{Z}}$, we have to pay a price in the form of a larger extension constant when using only the restricted compensation set $\mathcal{K}^{\complement}_{\mathbb{N}_0}$.

Appendix C
Technical Results

For $h \in H^1_0$ and $g \in H^\infty$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)g(\theta)\,\mathrm{d}\theta = 0.$$

Hence, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\theta)+h(\theta))g(\theta)\,\mathrm{d}\theta$$

for all $f \in L^1$, $g \in H^\infty$, and $h \in H^1_0$. This equality can be used to show that the dual space of (L^1/H^1_0) is given by $(L^1/H^1_0)^* = H^\infty$ [47, p. 198].

Thus, for any continuous linear functional Φ on L^1/H^1_0 there exists a function $g \in H^\infty$ such that

$$\Phi([f]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\mathrm{e}^{i\theta})\,\mathrm{d}\theta$$

for all $[f] \in L^1/H^1_0$. Further, we have

$$\|\Phi\|_\infty = \sup_{\substack{[f]\in L^1/H^1_0\\ \|[f]\|_{L^1/H^1_0}\leq 1}} |\Phi([f])| = \|g\|_{H^\infty}.$$

In the proof we need to reason with the space L^1/H^1_0, because for H^1 we have $(H^1)^* = \mathrm{BMOA}$, where BMOA denotes the space of functions in BMO that are analytic inside the unit circle.

We need some technical results about P_- that we will derive next. Let

$$E_\lambda(f) = \{\theta \in [-\pi,\pi] \colon |f(\theta)| > \lambda\}.$$

For $1 \leq p < \infty$ it is shown in [49, p. 7] that

$$\|f\|^p_{L^p} = \int_0^\infty p\lambda^{p-1}\mu(E_\lambda(f))\,\mathrm{d}\lambda. \tag{19}$$

However, we need this relation for $p = 1/2$, as stated in the following lemma.

Lemma 1. *We have*

$$\|f\|^{1/2}_{L^{1/2}} = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(f))\,\mathrm{d}\lambda. \tag{20}$$

Proof. For $p = 1$, we obtain from (19) that

$$\|g\|_{L^1} = \int_0^\infty \mu(E_\lambda(g))\,\mathrm{d}\lambda.$$

Let $g(\theta) = |f(\theta)|^{1/2}$. Then we have

$$\begin{aligned} E_\lambda(g) &= E_\lambda(|f|^{1/2}) \\ &= \{\theta \in [-\pi,\pi] \colon |f(t)|^{1/2} > \lambda\} \\ &= \{\theta \in [-\pi,\pi] \colon |f(t)| > \lambda^2\} \\ &= E_{\lambda^2}(f) \end{aligned}$$

and

$$\begin{aligned} \|f\|^{1/2}_{L^{1/2}} &= \|g\|_{L^1} \\ &= \int_0^\infty \mu(E_\lambda(g))\,\mathrm{d}\lambda \\ &= \int_0^\infty \mu(E_{\lambda^2}(f))\,\mathrm{d}\lambda. \end{aligned}$$

Let $\lambda_1 = \lambda^2$. Then it follows that

$$\int_0^\infty \mu(E_{\lambda^2}(f))\,\mathrm{d}\lambda = \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{\lambda_1}}\mu(E_{\lambda_1}(f))\,\mathrm{d}\lambda_1,$$

which completes the proof. □

We use Lemma 1 and apply it to the function P_-f to obtain the next lemma.

Lemma 2. *There exists a constant C_5 such that for all $f \in L^1$ we have*

$$\|P_-f\|_{L^{1/2}} \leq C_5\|f\|_{L^1}.$$

Proof. Using (16), we see that

$$\mu(\{\theta \in [-\pi,\pi] \colon |(P_-f)(\mathrm{e}^{i\theta}) > \lambda|\}) \leq \frac{C_4}{\lambda}\|f\|_{L^1}. \tag{21}$$

Further, according to Lemma 1, we have

$$\begin{aligned} \|P_-f\|^{1/2}_{L^{1/2}} &= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(P_-f))\,\mathrm{d}\lambda \\ &= \frac{1}{2}\int_0^{\|f\|_{L^1}} \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(P_-f))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{2}\int_{\|f\|_{L^1}}^\infty \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(P_-f))\,\mathrm{d}\lambda. \end{aligned}$$

We will analyze both integrals next. Since

$$\mu(E_\lambda(P_-f)) \leq 2\pi,$$

we obtain for the first integral

$$\begin{aligned} \frac{1}{2}\int_0^{\|f\|_{L^1}} \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(P_-f))\,\mathrm{d}\lambda &\leq \frac{2\pi}{2}\int_0^{\|f\|_{L^1}} \frac{1}{\sqrt{\lambda}}\,\mathrm{d}\lambda \\ &= 2\pi\left(\|f\|_{L^1}\right)^{\frac{1}{2}}. \end{aligned}$$

For the second integral we obtain, using (21), that

$$\begin{aligned} \frac{1}{2}\int_{\|f\|_{L^1}}^\infty \frac{1}{\sqrt{\lambda}}\mu(E_\lambda(P_-f))\,\mathrm{d}\lambda &\leq \frac{C_4\|f\|_{L^1}}{2}\int_{\|f\|_{L^1}}^\infty \frac{1}{\lambda\sqrt{\lambda}}\,\mathrm{d}\lambda \\ &= C_4\left(\|f\|_{L^1}\right)^{\frac{1}{2}}. \end{aligned}$$

Consequently, we have

$$\|P_-f\|_{L^{1/2}} \leq (2\pi + C_4)^2\|f\|_{L^1}.$$ □

Let

$$c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) = \inf_{\substack{f\in\overline{\mathfrak{F}}^1(\mathcal{K})\\ \|f\|_{L^1}=1}} \|[f]\|_{L^1/H^1_0}. \tag{22}$$

Clearly, we always have $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) \geq 0$. Moreover, the following lemma shows that $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) = 0$ cannot occur in the case that is relevant to us.

Lemma 3. *Let $\mathcal{K} \subset \mathbb{N}_0$ and assume that there exits a constant C_1 such that*

$$\|f\|_{L^2} \leq C_1 \|f\|_{L^1} \tag{23}$$

for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Then we have $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) > 0$.

Proof. We prove the lemma by contradiction, and assume that $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) = 0$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \overline{\mathfrak{F}}^1(\mathcal{K})$ with $\|f_n\|_{L^1} = 1$ such that

$$\lim_{n\to\infty} \|[f_n]\|_{L^1/H_0^1} = 0.$$

For every $n \in \mathbb{N}$ there exists exactly one $h_n \in H_0^1$ such that

$$\|[f_n]\|_{L^1/H_0^1} = \|f_n + h_n\|_{L^1}.$$

Since $h_n \in H_0^1$, we have $P_-(h_n) = 0$, and it follows that $P_-(f_n + h_n) = f_n$. Hence, Lemma 2 implies that

$$\|f_n\|_{L^{1/2}} \leq C_5 \|f_n + h_n\|_{L^1} = C_5 \|[f_n]\|_{L^1/H_0^1},$$

and we obtain

$$\lim_{n\to\infty} \|f_n\|_{L^{1/2}} = 0. \tag{24}$$

According to the definition of the sequence $\{f_n\}_{n\in\mathbb{N}}$ we have $\|f_n\|_{L^1} = 1$, and due to (23) we obtain $\|f_n\|_{L^2} \leq C_1$. Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)| \,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)|^{\frac{1}{4}} |f_n(\theta)|^{1-\frac{1}{4}} \,\mathrm{d}\theta \\
&\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)|^{\frac{1}{2}} \,\mathrm{d}\theta\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)|^{\frac{3}{2}} \,\mathrm{d}\theta\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)|^{\frac{1}{2}} \,\mathrm{d}\theta\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(\theta)|^{2} \,\mathrm{d}\theta\right)^{\frac{1}{2}\frac{3}{4}} \\
&\leq \|f_n\|_{L^{1/2}}^{\frac{1}{4}} C_1^{\frac{3}{4}},
\end{aligned}$$

where we used Hölder's inequality in the second to last line, and consequently

$$\|f_n\|_{L^{1/2}} \geq C_1^{-3},$$

which is a contradiction to (24). Hence, our assumption $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) = 0$ was wrong, and it follows $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) > 0$. □

Appendix D
Proof of the Main Result

We split the proof of Theorem 2 into two parts, the "⇒" and the "⇐" direction. The statement of the "⇐" direction is rephrased in the following theorem.

Theorem 8. *Let $\mathcal{K} \subset \mathbb{N}_0$ be such that there exists a constant C_1 such that*

$$\|f\|_{L^2} \leq C_1 \|f\|_{L^1} \tag{25}$$

for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Then, for each $a \in \ell^2(\mathcal{K})$, there exists a $g \in H^\infty$ such that

$$\|g\|_{H^\infty} \leq \frac{C_1}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))} \|a\|_{\ell^2(\mathcal{K})}$$

and

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta, \quad k \in \mathcal{K}.$$

For the proof of Theorem 8 we need one further lemma. Let $\overline{\mathfrak{F}}^1(\mathcal{K})$ be the Banach space that was defined in (18), and define the linear operator T by

$$T\colon \overline{\mathfrak{F}}^1(\mathcal{K}) \to L^1/H_0^1, f \mapsto [f] = \{f + h\}_{h\in H_0^1}.$$

Since

$$\|Tf\|_{L^1/H_0^1} = \inf_{g\in H_0^1} \|f + g\|_{L^1} \leq \|f + 0\|_{L^1} = \|f\|_{L^1},$$

we see that T is a bounded operator. Further, let

$$B_1 = T[\overline{\mathfrak{F}}^1(\mathcal{K})]$$

be the image of $\overline{\mathfrak{F}}^1(\mathcal{K})$ under T.

Lemma 4. *Let $\mathcal{K} \subset \mathbb{N}_0$ and assume that there exits a constant C_1 such that*

$$\|f\|_{L^2} \leq C_1 \|f\|_{L^1}$$

for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Then $B_1 = T[\overline{\mathfrak{F}}^1(\mathcal{K})]$ is a closed subspace of L^1/H_0^1.

Proof. T is a bounded linear operator. Hence, B_1 is a vector space. It remains to show that B_1 is complete. Let $\{[f_n]\}_{n\in\mathbb{N}}$ be a Cauchy sequence in B_1. Using $B_1 \subset L^1/H_0^1$, it follows that $\{[f_n]\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in L^1/H_0^1. Since L^1/H_0^1 is a Banach space, there exists an $[f_*] \in L^1/H_0^1$ such that

$$\lim_{n\to\infty} \|[f_n] - [f_*]\|_{L^1/H_0^1} = 0.$$

From the definition of c_1 in (22) we see that $\|Tf\|_{L^1/H_0^1} \geq c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))$ for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$ with $\|f\|_{L^1} = 1$, and consequently that

$$\|Tf\|_{L^1/H_0^1} \geq c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))\|f\|_{L^1}$$

for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Since $c_1(\overline{\mathfrak{F}}^1(\mathcal{K})) > 0$, according to Lemma 3, we obtain

$$\frac{\|[f]\|_{L^1/H_0^1}}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))} = \frac{\|Tf\|_{L^1/H_0^1}}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))} \geq \|f\|_{L^1} \tag{26}$$

for all $f \in \overline{\mathfrak{F}}^1(\mathcal{K})$. Thus, we have

$$\frac{\|[f_n] - [f_m]\|_{L^1/H_0^1}}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))} \geq \|f_n - f_m\|_{L^1},$$

which shows that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{\mathfrak{F}}^1(\mathcal{K})$. Hence, there exists a $g_* \in \overline{\mathfrak{F}}^1(\mathcal{K})$ such that

$$\lim_{n\to\infty} \|f_n - g_*\|_{L^1} = 0.$$

We have

$$\begin{aligned}\|[f_n]-[g_*]\|_{L^1/H^1_0} &= \|T(f_n-g_*)\|_{L^1/H^1_0}\\ &\leq \|f_n-g_*\|_{L^1},\end{aligned}$$

and consequently

$$\lim_{n\to\infty}\|[f_n]-[g_*]\|_{L^1/H^1_0}=0.$$

It follows that $[g_*]=[f_*]$, which shows that $[f_*]\in B_1$. Thus, B_1 is complete. □

Now we are in the position to prove Theorem 8.

Proof of Theorem 8: For $a\in\ell^2(\mathcal{K})$ and $f\in B_1$ we define the linear functional

$$\Phi_a([f])=\sum_{k\in\mathcal{K}}c_{-k}(f)a_k.$$

Φ_a is well-defined, because $[f]=[g]$ implies $g=f+h$ for some $h\in H^1_0$, and we have

$$\begin{aligned}\Phi_a([g]) &= \Phi_a([f+h])\\ &= \sum_{k\in\mathcal{K}}a_k\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(\theta)+h(\theta))\,\mathrm{e}^{ik\theta}\;\mathrm{d}\theta\\ &= \sum_{k\in\mathcal{K}}a_k\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,\mathrm{e}^{ik\theta}\;\mathrm{d}\theta\\ &= \sum_{k\in\mathcal{K}}c_{-k}(f)a_k\\ &= \Phi_a([f]).\end{aligned}$$

Let $[f]\in B_1=T[\overline{\mathfrak{F}}^1(\mathcal{K})]$ be arbitrary but fixed. We have, using the assumption (25), that

$$\begin{aligned}|\Phi_a([f])| &\leq \left(\sum_{k\in\mathcal{K}}|c_{-k}(f)|^2\right)^{\frac{1}{2}}\left(\sum_{k\in\mathcal{K}}|a_k|^2\right)^{\frac{1}{2}}\\ &= \|f\|_{L^2}\|a\|_{\ell^2(\mathcal{K})}\\ &\leq C_1\|f\|_{L^1}\|a\|_{\ell^2(\mathcal{K})}.\end{aligned}\tag{27}$$

As shown in (26), we also have

$$\frac{\|[f]\|_{L^1/H^1_0}}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))}\geq\|f\|_{L^1}.$$

Combining (26) and (27), it follows that

$$|\Phi_a([f])|\leq\frac{C_1}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))}\|[f]\|_{L^1/H^1_0}\|a\|_{\ell^2(\mathcal{K})}.$$

Thus, the functional $\Phi_a\colon B_1\to\mathbb{C}$ is a continuous linear functional on B_1, and its norm satisfies

$$\|\Phi_a\|_{B_1\to\mathbb{C}}\leq\frac{C_1}{c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))}\|a\|_{\ell^2(\mathcal{K})}.$$

From Lemma 4 we know that B_1 is a closed subspace of L^1/H^1_0. According to the Hahn–Banach theorem [50, p. 104, Theorem 5.16] we can extend the functional Φ_a to a continuous linear functional Φ_a^{Ex} on whole of L^1/H^1_0, while keeping the norm of Φ_a. Since the dual space of L^1/H^1_0 is H^∞, it follows that there exists a $g_{\mathrm{Ex}}\in H^\infty$, such that

$$\Phi_a^{\mathrm{Ex}}([f])=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)g_{\mathrm{Ex}}(\theta)\;\mathrm{d}\theta\tag{28}$$

for all $[f]\in L^1/H^1_0$. As above, Φ_a^{Ex} is well defined, because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}h(\theta)g_{\mathrm{Ex}}(\theta)\;\mathrm{d}\theta=0$$

for all $h\in H^1_0$. We have $\|\Phi_a^{\mathrm{Ex}}\|=\|g_{\mathrm{Ex}}\|_{H^\infty}$. For $k\in\mathcal{K}$ and

$$f_k(\theta)=\mathrm{e}^{-ik\theta},\quad\theta\in[-\pi,\pi],$$

we have $f_k\in\overline{\mathfrak{F}}^1(\mathcal{K})$ and $[f_k]\in B_1$. It follows that

$$\Phi_a^{\mathrm{Ex}}([f_k])=\Phi_a([f_k])=\sum_{l\in\mathcal{K}}a_l\frac{1}{2\pi}\int_{-\pi}^{\pi}f_k(\theta)\,\mathrm{e}^{il\theta}\;\mathrm{d}\theta=a_k$$

for all $k\in\mathcal{K}$. From (28) we see that

$$\Phi_a^{\mathrm{Ex}}([f_k])=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-ik\theta}\,g_{\mathrm{Ex}}(\theta)\;\mathrm{d}\theta=c_k(g_{\mathrm{Ex}}).$$

Thus, g_{Ex} satisfies $c_k(g_{\mathrm{Ex}})=a_k$ for all $k\in\mathcal{K}$, $c_k(g_{\mathrm{Ex}})=0$ for all $k<0$, and $\|g_{\mathrm{Ex}}\|_{H^\infty}=\|\Phi_a^{\mathrm{Ex}}\|=\|\Phi_a\|\leq C_1/c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))$. That is, g_{Ex} solves the OFDM CF problem with restricted carrier set with extension constant $C_1/c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))$. ■

We have seen that the condition (25) is sufficient for the strong solvability of the CF problem with restricted carrier set. The necessity of the condition (25) for the strong solvability of the CF problem with restricted carrier set is easy to see and stated in the next theorem.

Theorem 9. *Let $\mathcal{K}\subset\mathbb{N}_0$. If the OFDM CF problem with restricted carrier set ($\mathcal{I}=\mathbb{N}_0$) is strongly solvable then there exists a constant C_1 such that* (25) *holds for all $f\in\overline{\mathfrak{F}}^1(\mathcal{K})$.*

Proof: Let $\mathcal{K}\subset\mathbb{N}_0$. If the CF problem is strongly solvable when using only the restricted compensation set, $\mathcal{K}^{\complement}_{\mathbb{N}_0}$ then it is also strongly solvable when using the full compensation set $\mathcal{K}^{\complement}_{\mathbb{Z}}$. Hence, Theorem 1 implies that there exists a finite constant C_1 such that (25) holds for all $f\in\overline{\mathfrak{F}}^1(\mathcal{K})$. ■

Theorems 8 and 9 together imply Theorem 2.

For the CF problem with full carrier set ($\mathcal{I}=\mathbb{Z}$), we know that the optimal, i.e. smallest extension constant $\underline{C}^{\mathbb{Z}}_{\mathrm{Ex}}$ is the smallest constant for which (25) holds for all $f\in\overline{\mathfrak{F}}^1(\mathcal{K})$. As discussed above, for the optimal extension constant in the CF problem with restricted carrier set ($\mathcal{I}=\mathbb{N}_0$) we only have the upper bound $C_1/c_1(\overline{\mathfrak{F}}^1(\mathcal{K}))$. We conjecture that the optimal extension constant for the CF problem with restricted carrier set is in general much larger than the optimal extension constant for the CF problem with full carrier set.

Appendix E
Proof of Theorem 6

Proof of Theorem 6. The proof is almost identical to the proof of Theorem 1 in [46]. We only discuss the differences here. Let

$$H^\infty_{\mathcal{K}}=\left\{f\in H^\infty\colon\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\,\mathrm{e}^{-ik\theta}\;\mathrm{d}\theta=0,\;k\in\mathcal{K}\right\}.$$

Then $H^\infty_{\mathcal{K}}$ is a closed subspace of H^∞. The rest of the proof follows along the lines of the proof of Theorem 1 in [46], when the quotient space $Q_{\mathcal{K}}=H^\infty/H^\infty_{\mathcal{K}}$ is considered. □

Appendix F
Proof of Theorem 7

Proof of Theorem 7. Let

$$\mathcal{Z}_M = \left\{ a \in \ell^2(\mathcal{K}) \colon \exists f \in H^\infty, \|f\|_{H^\infty} \leq M \right.$$
$$\left. \text{with } \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = a_k,\, k \in \mathcal{K} \right\}.$$

We have

$$\mathcal{D}_{\mathbb{N}_0}^{\complement} = \ell^2(\mathcal{K}) \setminus \mathcal{D}_{\mathbb{N}_0} = \bigcup_{M \in \mathbb{N}} \mathcal{Z}_M.$$

Assume that the CF problem with restricted carrier set is not weakly solvable. Then there exists an $a \in \ell^2(\mathcal{K})$ such that $\mathcal{B}_{\mathbb{N}_0}(a) = \emptyset$. We will show that

$$\mathcal{D}_{\mathbb{N}_0}^{\complement} = \{a \in \ell^2(\mathcal{K}) \colon \mathcal{B}_{\mathbb{N}_0}(a) \neq \emptyset\}$$

is a set of first category. According to the definition of a residual set, this implies that $\mathcal{D}_{\mathbb{N}_0}$ is a residual set.

We prove that, for all $M \in \mathbb{N}_0$, the set $\mathcal{Z}_M$ is nowhere dense in $\ell^2(\mathcal{K})$. Then it follows that $\mathcal{D}_{\mathbb{N}_0}^{\complement}$, as the countable union of nowhere dense sets, is a set of first category.

We do a proof by contradiction: We assume that there exists an $M_0 \in \mathbb{N}$ such that $\mathcal{Z}_{M_0}$ is not nowhere dense, and show that this assumption leads to a contradiction. According to the assumption there exist an $\hat{a} \in \ell^2(\mathcal{K})$ and a $\delta > 0$ such that

$$\mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$$

is dense in $B_\delta(\hat{a})$, where

$$B_\delta(\hat{a}) = \{a \in \ell^2(\mathcal{K}) \colon \|a - \hat{a}\|_{\ell^2(\mathcal{K})} < \delta\}$$

denotes the open ball at $\hat{a}$ with radius δ.

Let $a \in B_\delta(\hat{a})$ be arbitrary. Since $\mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$ is dense in $B_\delta(\hat{a})$, there exists a sequence $\{a^{(N)}\}_{N\in\mathbb{N}} \subset \mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$ such that

$$\lim_{N\to\infty} \|a - a^{(N)}\|_{\ell^2(\mathcal{K})} = 0.$$

Further, for every $N \in \mathbb{N}$, there exists an $f_N \in H^\infty(\mathbb{D})$ with $\|f_N\|_{H^\infty} \leq M_0$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_N(\mathrm{e}^{i\theta})\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = a_k^{(N)}, \quad k \in \mathcal{K}.$$

According to Montel's theorem [51, p. 195] there exists a subsequence $\{N_n\}_{n\in\mathbb{N}}$ and an f_* that is analytical in $\mathbb{D}$, such that

$$\lim_{n\to\infty} f_{N_n}(z) = f_*(z), \tag{29}$$

where the convergence is uniform on compact subsets of $\mathbb{D}$. Thus, for all $z \in \mathbb{D}$ we have

$$|f_*(z)| = \lim_{n\to\infty} |f_{N_n}(z)| \leq M_0,$$

which implies that $f_* \in H^\infty(\mathbb{D})$. We have

$$f_{N_n}(z) = \sum_{l=0}^{\infty} c_l(f_{N_n}) z^l, \quad z \in \mathbb{D},$$

where

$$\begin{aligned} c_l(f_{N_n}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{N_n}(\mathrm{e}^{i\theta})\,\mathrm{e}^{-il\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{N_n}(\rho\,\mathrm{e}^{i\theta})\,\mathrm{e}^{-il\theta}\,\mathrm{d}\theta \rho^{-l}, \quad l, n \in \mathbb{N}, \end{aligned}$$

for $0 < \rho \leq 1$. The same holds for f_*. Using (29), it follows that

$$\lim_{n\to\infty} c_l(f_{N_n}) = c_l(f_*).$$

In particular, we have for $k \in \mathcal{K}$ that

$$c_k(f_*) = \lim_{n\to\infty} a_k^{(N_n)} = a_k.$$

Hence, we see that $a \in \mathcal{Z}_{M_0}$. Since $a \in B_\delta(\hat{a})$ was arbitrary, it follows that $B_\delta(\hat{a}) \subset \mathcal{Z}_{M_0}$, which also implies that $\hat{a} \in \mathcal{Z}_{M_0}$.

According to the assumption of the theorem, the CF is not weakly solvable. Hence, there exists an $\tilde{a} \in \ell^2(\mathcal{K})$ with $\mathcal{B}_{\mathbb{N}_0}(\tilde{a}) = \emptyset$. We set

$$\alpha := \hat{a} + \frac{\delta}{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}} \tilde{a} \in B_\delta(\hat{a}) \subset \mathcal{Z}_{M_0}.$$

Thus, there must exist an $f_1 \in H^\infty$, $\|f_1\|_{H^\infty} \leq M_0$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = \alpha_k, \quad k \in \mathcal{K}.$$

Since $\hat{a} \in \mathcal{Z}_{M_0}$, there exists an $f_2 \in H^\infty$, $\|f_2\|_{H^\infty} \leq M_0$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = \hat{a}_k, \quad k \in \mathcal{K}.$$

For

$$f_3 := \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta}(f_1 - f_2)$$

we have

$$\begin{aligned} \|f_3\|_{H^\infty} &= \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta} \|f_1 - f_2\|_{H^\infty} \\ &\leq \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta} (\|f_1\|_{H^\infty} + \|f_2\|_{H^\infty}) \\ &\leq \frac{4\|\tilde{a}\|_{\ell^2(\mathcal{K})} M_0}{\delta}, \end{aligned}$$

which shows that $f_3 \in H^\infty$. For $k \in \mathcal{K}$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_3(\theta)\,\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta = \frac{2}{\delta}(\alpha_k - \hat{a}_k) = \tilde{a}_k, \quad k \in \mathcal{K}.$$

Therefore, we have $\tilde{a} \in \mathcal{Z}_{\tilde{M}}$, where $\tilde{M}$ is the smallest natural number such that $\tilde{M} \geq 4\|\tilde{a}\|_{\ell^2(\mathcal{K})} M_0/\delta$. It follows that $\mathcal{B}_{\mathbb{N}_0}(\tilde{a}) \neq \emptyset$, which is a contradiction. □

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