ANALYTIC PROPERTIES OF DOWNSAMPLING FOR BANDLIMITED SIGNALS

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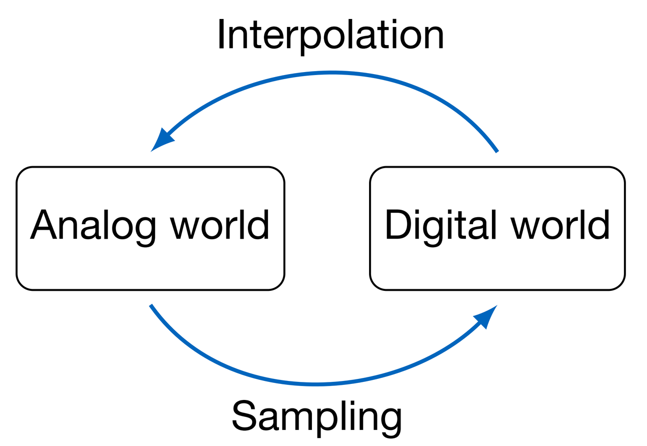
Introduction

- We study downsampling and bandlimited interpolation for bandlimited signals.
- In signal processing books: the theoretical treatment of downsampling and bandlimited interpolation is not given special attention, despite their high importance in applications.
- Conception: the bandlimited interpolation exists always.
- We construct a bandlimited signal, which after downsampling does not have a bounded bandlimited interpolation.
  $\Rightarrow$ downsampling needs to be treated carefully

Motivation

"Equivalence" between analog and digital world



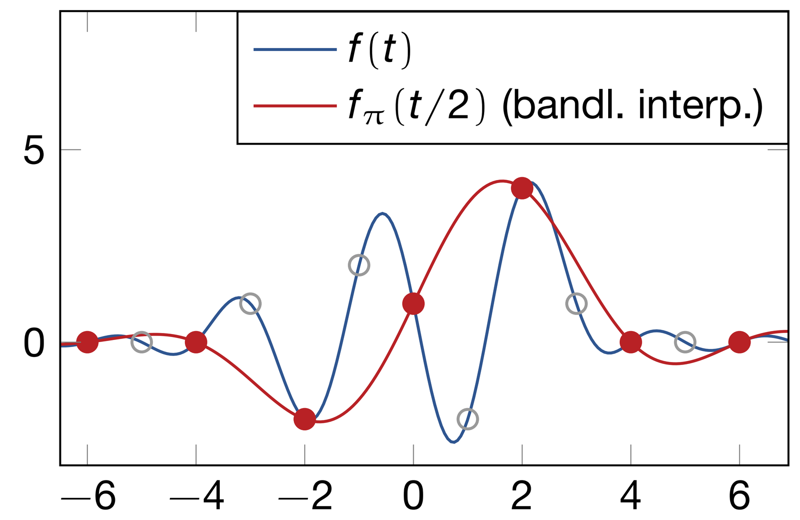
Sampling: $f(t) \to$ sampled signal is $\{x_k\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}}$

Downsampling: Process of reducing the sampling rate of a discrete-time signal by removing samples.

$\{x_k\}_{k\in\mathbb{Z}} \to$ downsampled signal is $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{x_{2k}\}_{k\in\mathbb{Z}}$

Bandlimited interpolation:
Find a signal f_π with bandwidth π that interpolates the downsampled signal $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}}$, i.e., satisfies:

$$f_\pi(k) = x_k^{\text{down}}, \qquad k \in \mathbb{Z}$$



We study the existence of the bandlimited interpolation for sequences that are created by downsampling a discrete-time signal that has been generated by sampling a bandlimited signals.

Notation

$L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$: the usual L^p-spaces. $\ell^2(\mathbb{Z})$: set of all square summable sequences. c_0: set of all sequences that vanish at infinity. $C_0^\infty[0,1]$: space of all functions that have continuous derivatives of all orders and are zero outside $[0,1]$.
Bernstein space $\mathcal{B}_\sigma^p$ ($\sigma > 0$, $1 \leqslant p \leqslant \infty$): space of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$. Norm: L^p-norm on the real line.
A signal in $\mathcal{B}_\sigma^p$ is bandlimited to σ. $\mathcal{B}_\sigma^2$ is the frequently used space of bandlimited functions with bandwidth σ and finite energy. We call a signal in $\mathcal{B}_\pi^\infty$ bounded bandlimited signal. $\mathcal{B}_{\sigma,0}^\infty$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.

Downsampling and Bandlimited Interpolation

Signals in $\mathcal{B}_{2\pi}^2$ (bandlimited, finite energy)

- $f \in \mathcal{B}_{2\pi}^2$ is completely determined by its samples $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}}$. We have

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2}\right) \frac{\sin(2\pi(t-\frac{k}{2}))}{2\pi(t-\frac{k}{2})} \right| = 0.$$

- Downsampling: We have $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.
- Bandlimited interpolation: $f_\pi \in \mathcal{B}_\pi^2$ exists and is given by

$$f_\pi(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

For $\mathcal{B}_{2\pi}^2$ downsampling and bandlimited interpolation are well-behaved. Equivalence between continuous-time and discrete-time is preserved.

Signals in $\mathcal{B}_{2\pi,0}^\infty$ (bandlimited, bounded, vanish at infinity)

- $f \in \mathcal{B}_{2\pi,0}^\infty$ is uniquely determined by its samples $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}}$. For all $T > 0$ we have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2}\right) \frac{\sin(2\pi(t-\frac{k}{2}))}{2\pi(t-\frac{k}{2})} \right| = 0.$$

- Downsampling: We have $\{f(\frac{k}{2})\}_{k\in\mathbb{Z}} \in c_0$ and $\{x_k^{\text{down}}\}_{k\in\mathbb{Z}} = \{f(k)\}_{k\in\mathbb{Z}} \in c_0$.

Question: Is there a continuous-time signal $f_\pi \in \mathcal{B}_\pi^\infty$ that interpolates $\{f(k)\}_{k\in\mathbb{Z}}$?

Distributional Behavior

In many books the bandlimited interpolation is formally obtained by using a convolution theorem and distribution theory.

1. The discrete-time signal is created by multiplying f with a Dirac comb

$$f_{\text{III}}(t) = f(t) \cdot \text{III}(t) = f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{k=-\infty}^{\infty} f(k)\delta(t-k).$$

2. The bandlimited interpolation is obtained by convolving f_{III} with the impulse response of the ideal low-pass filter

$$f_\pi(t) = (f_{\text{III}} * \text{sinc})(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

It is not clear whether the above manipulations and expressions are always well-defined.

Another example where even the theory of distributions fails are convolution sum system representations.

H. Boche, U. Mönich, and B. Meinerzhagen, "Non-existence of convolution sum system representations," *IEEE Trans. Signal Process.*, vol. 67, no. 10, pp. 2649–2664, May 2019

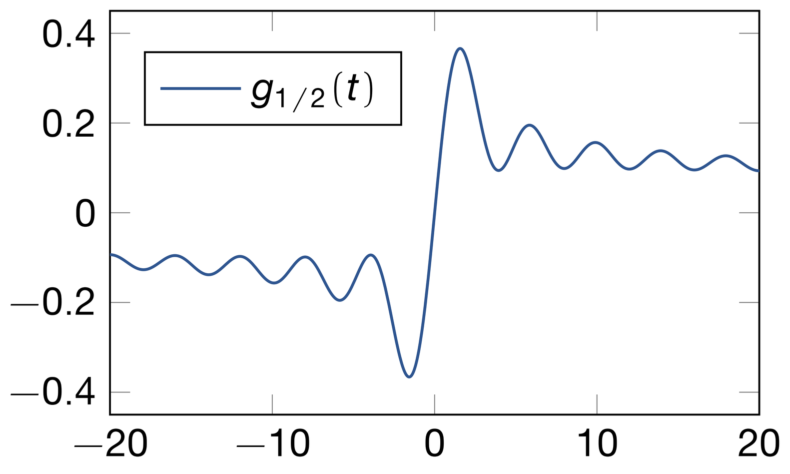
Main Result

We use the signal

$$\gamma_\delta(t) = e^{i\pi t} g_\delta(t)$$

where

$$g_\delta(t) = \frac{1}{\pi} \int_0^{\delta\pi} \frac{\sin(\omega t)}{\omega \log(\frac{\pi}{\omega})} \, d\omega.$$



- γ_δ is a bandpass signal that is created by modulating the lowpass signal g_δ.
- The spectrum of the lowpass signal g_δ is concentrated on $[-\delta\pi, \delta\pi]$.
- We have $\gamma_\delta \in \mathcal{B}_{(1+\delta)\pi,0}^\infty \subset \mathcal{B}_{2\pi,0}^\infty$ (the effective bandwidth of γ_δ is $2\delta\pi$).

Theorem: *Let $\delta \in (0,1)$. There exists no $f_\pi \in \mathcal{B}_\pi^\infty$ with $f_\pi(k) = \gamma_\delta(k)$ for all $k \in \mathbb{Z}$. That is, there exists no bounded bandlimited interpolation for the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$.*

For the downsampled sequence $\{\gamma_\delta(k)\}_{k\in\mathbb{Z}}$, the Shannon sampling series diverges (even in a distributional setting).

Theorem: *Let $\delta \in (0,1)$. Then, for all $t \in \mathbb{R} \setminus \mathbb{Z}$, we have*

$$\lim_{N\to\infty} \left| \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty.$$

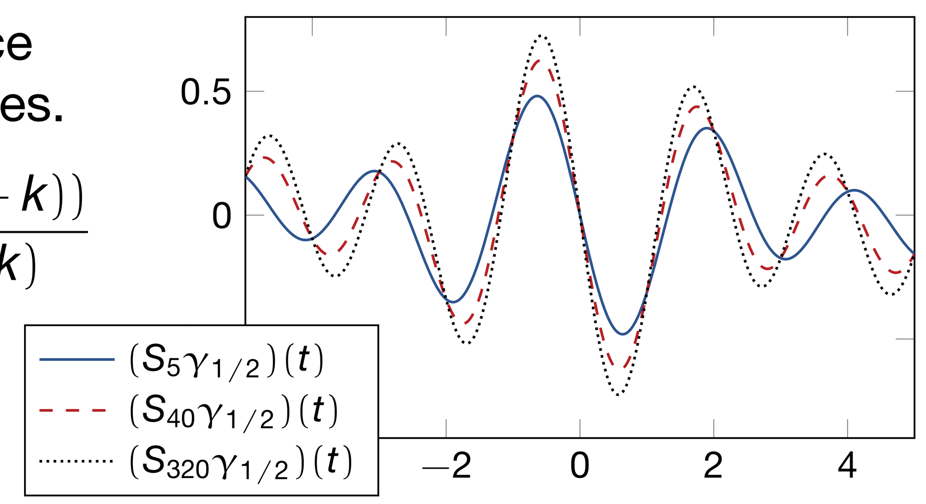
Further, there exists a $\phi_1 \in C_0^\infty[0,1]$ such that

$$\lim_{N\to\infty} \left| \int_{-\infty}^{\infty} \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \phi_1(t) \, dt \right| = \infty,$$

i.e., the series diverges in $\mathcal{D}'$.

Visualization of the divergence of the Shannon sampling series.

$$(S_N \gamma_\delta)(t) = \sum_{k=-N}^{N} \gamma_\delta(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$



It is well-known that there exist sequences that do not possess a bounded bandlimited interpolation.
Example:

$$x_k = \begin{cases} 0, & k \leqslant 0, \\ \frac{(-1)^k}{\log(1+k)}, & k \geqslant 1. \end{cases}$$

Note: The situation here is more complicated. The sequence is not freely chosen but obtained by downsampling of a bounded bandlimited signal.