Effective Approximation of Bandlimited Signals and Their Samples

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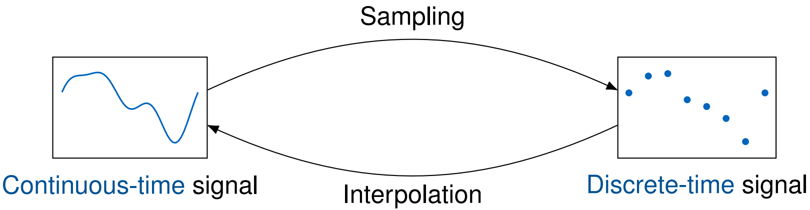
Motivation

Shannon's sampling theorem links the continuous-time and discrete-time worlds.



- When applying sampling or interpolation, many properties and characteristics of the signal carry over from one domain into the other (e.g., energy in discrete-time = energy in continuous-time).
- We analyze if and how this transition affects the computability of the signal.

Why Study the Computability of a Signal?

- In many applications digital hardware is used (CPUs, FPGAs, etc.).
- Computability of a signal is directly linked to the approximation with “simple” signals, where we have an “effective”/algorithmic control of the approximation error.
- If a signal is not computable, we cannot control the approximation error.

Overview of the Results

We study bandlimited signals $f \in \mathcal{B}_\pi^p$ with finite L^p-norm.

Computability continuous-time $\overset{?}{\Leftrightarrow}$ computability discrete-time

$p \in (1, \infty)$	$p = 1$ or $p = \infty$
✓ Correspondence	x No correspondence
Algorithm: Shannon sampling series	No algorithm exists
Control of the approximation error	No control of the approximation error

Turing Machine

Turing Machine:
Abstract device that manipulates symbols on a strip of tape according to certain rules.

- Turing machines are an idealized computing model.
- No limitations on computing time or memory, no computation errors.
- Although the concept is very simple, Turing machines are capable of simulating any given algorithm.

Turing machines are suited to study the limitations of a digital computer:

> Anything that is not Turing computable cannot be computed on a real digital computer, regardless how powerful it may be.

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem," *Proceedings of the London Mathematical Society*, vol. s2-42, no. 1, pp. 230–265, Nov. 1936

A. M. Turing, "On computable numbers, with an application to the Entscheidungsproblem. A correction," *Proceedings of the London Mathematical Society*, vol. s2-43, no. 1, pp. 544–546, Jan. 1937

Notation

- c_0: space of all sequences that vanish at infinity
- $\ell^p(\mathbb{Z})$, $1 \leqslant p < \infty$: spaces of p-th power summable sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$
  Norm: $\|x\|_{\ell^p} = (\sum_{k=-\infty}^{\infty} |x(k)|^p)^{1/p}$
- $L^p(\Omega)$, $1 \leqslant p < \infty$: space of all measurable, pth-power Lebesgue integrable functions on Ω
  Norm: $\|f\|_p = \left(\int_\Omega |f(t)|^p \, dt\right)^{1/p}$
- $L^\infty(\Omega)$: space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite

Bandlimited Functions

Definition (Bernstein Space)

Let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leqslant C(\epsilon) \exp\big((\sigma + \epsilon)|z|\big)$ for all $z \in \mathbb{C}$.

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line.

- A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ.
- We have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r$ for all $1 \leqslant p \leqslant r \leqslant \infty$.
- $\mathcal{B}_{\sigma,0}^\infty$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.
- $\mathcal{B}_\sigma^2$: space of bandlimited functions with finite energy.

Computable Sequences of Rationals

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

- A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.

Computable Real Numbers

First example of an effective approximation

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|x - r_n| < 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathbb{R}_c$: set of computable real numbers
- $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable.
- Commonly used constants like e and π are computable.

Computability in ℓ^p

Computability in Banach spaces: Effective approximation by "simple" elements

A sequence $x = \{x(k)\}_{k\in\mathbb{Z}}$ in ℓ^p, $p \in [1,\infty) \cap \mathbb{R}_c$ is called computable in ℓ^p if every number $x(k)$, $k \in \mathbb{Z}$, is computable and there exist a computable sequence $\{y_n\}_{n\in\mathbb{N}} \subset \ell^p$, where each y_n has only finitely many non-zero elements, all of which are computable as real numbers, and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have

$$\|x - y_n\|_{\ell^p} \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- Effective approximation by simple / finite-length sequences
- $\mathcal{C}\ell^p$: set of all sequences that are computable in ℓ^p
- $\mathcal{C}c_0$: set of all sequences that are computable in c_0

Computable Bandlimited Functions I

We call a function f elementary computable if there exists a natural number L and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

- Every elementary computable function is Turing computable.
- For every elementary computable function f, the norm $\|f\|_{\mathcal{B}_\pi^p}$ is computable.

Computable Bandlimited Functions II

A function in $f \in \mathcal{B}_\pi^p$, $1 \leqslant p < \infty$, is called computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_p \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathcal{CB}_\pi^p$: set of all functions that are computable in $\mathcal{B}_\pi^p$.
- $\mathcal{CB}_{\pi,0}^\infty$: set of all functions that are computable in $\mathcal{B}_{\pi,0}^\infty$ (analog definition).

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We can approximate every function $f \in \mathcal{CB}_\pi^p$ by an elementary computable function, where we have an effective control of the approximation error.

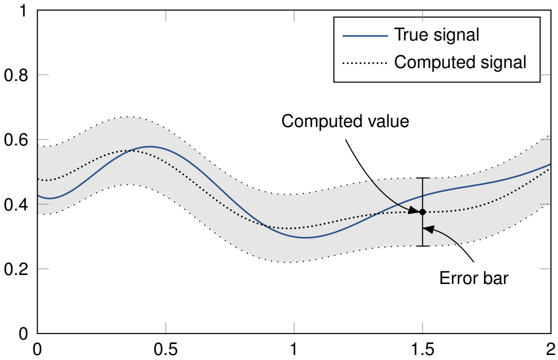
Computable Bandlimited Functions III

For $f \in \mathcal{CB}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$ and all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_\infty \leqslant (1 + \pi)\|f - f_n\|_p \leqslant \frac{1 + \pi}{2^M}$$

for all $n \geqslant \xi(M)$.

We can approximate any function $f \in \mathcal{CB}_\pi^p$ by an elementary computable function, where we have an effective and uniform control of the approximation error.



Observation

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Let $f \in \mathcal{CB}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, or $f \in \mathcal{CB}_{\pi,0}^\infty$.
Then $f|_\mathbb{Z} = \{f(k)\}_{k \in \mathbb{N}}$ is a computable sequence of computable numbers.
Further we have $f|_\mathbb{Z} \in \mathcal{C}\ell^p$ if $p \in [1, \infty) \cap \mathbb{R}_c$, and $f|_\mathbb{Z} \in \mathcal{C}c_0$ if $p = \infty$.

Continuous-time signal f computable $\Rightarrow$ Discrete-time signal $f|_\mathbb{Z}$ computable

Two Questions

Given a computable discrete-time signal, is the corresponding continuous-time signal computable?

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Question 1:
Is there a simple necessary and sufficient condition for characterizing the computability of f?

Two Questions

Given a computable discrete-time signal, is the corresponding continuous-time signal computable?

Continuous-time signal f computable $\not\Leftarrow$ Discrete-time signal $f|_{\mathbb{Z}}$ computable

Question 1:
Is there a simple necessary and sufficient condition for characterizing the computability of f?

Question 2:
Is there a simple canonical algorithm to actually compute f from the samples $f|_{\mathbb{Z}}$?

A Necessary and Sufficient Condition

Theorem

Let $f \in \mathcal{B}_{\pi}^{p}$, $p \in (1, \infty) \cap \mathbb{R}_c$. *Then we have* $f \in \mathcal{C}\mathcal{B}_{\pi}^{p}$ *if and only if* $f|_{\mathbb{Z}} \in \mathcal{C}\ell^{p}$.

- For $p \in (1, \infty) \cap \mathbb{R}_c$, the computability of the discrete-time signal implies the computability of the continuous-time signal.
- This answers Question 1.

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- For $p \in (1, \infty) \cap \mathbb{R}_c$, the computability of the discrete-time signal implies the computability of the continuous-time signal.
- This answers Question 1.

For $p \in (1, \infty) \cap \mathbb{R}_c$ we have: f computable $\Leftrightarrow$ $f|_\mathbb{Z}$ computable

That is we have a correspondence between the computable discrete-time signals in $\mathcal{C}\ell^p$ and the computable continuous-time signals in $\mathcal{CB}_\pi^p$.

No correspondence for $p = 1$

For $p = 1$ we do not have the correspondence. There exist signals that are in $\mathcal{C}\ell^1$ (computable in discrete-time), where the corresponding continuous-time signal is not in $\mathcal{CB}^1_\pi$.

Example:

- $f_1(t) = \sin(\pi t)/(\pi t)$, $t \in \mathbb{R}$.
- f_1 is a function of exponential type at most π and we have $f_1|_{\mathbb{Z}} \in \mathcal{C}\ell^1$.
- However, $f_1 \notin \mathcal{CB}^1_\pi$, because $f_1 \notin \mathcal{B}^1_\pi$.

No correspondence for $p = \infty$

For $p = \infty$ we also do not have the correspondence.

Theorem

There exists a $f_2 \in \mathcal{B}^\infty_{\pi,0}$ *such that* $f_2|_{\mathbb{Z}} \in \mathcal{C}c_0$ *and* $f_2 \notin \mathcal{C}\mathcal{B}^\infty_{\pi,0}$.
(We even have $f_2(t) \notin \mathbb{C}_c$ *for all* $t \in \mathbb{R}_c \setminus \mathbb{Z}$*).*

A Further Necessary and Sufficient Condition

Theorem

Let $f \in \mathcal{B}_{\pi}^{p}$, $p \in (1, \infty) \cap \mathbb{R}_c$. *We have* $f \in \mathcal{C}\mathcal{B}_{\pi}^{p}$ *if and only if*

1. $f|_{\mathbb{Z}}$ *is a computable sequence of computable numbers,*
2. $\|f|_{\mathbb{Z}}\|_{\ell^p} \in \mathbb{R}_c$.

- We do not require that the sequence $f|_{\mathbb{Z}}$ is computable in ℓ^p, but only that the number $\|f|_{\mathbb{Z}}\|_{\ell^p}$ is computable.

An Answer to Question 2

Shannon sampling series

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

Theorem

Let $p \in (1, \infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$. Then we have $f \in \mathcal{C}\mathcal{B}_\pi^p$ if and only if $f|_\mathbb{Z}$ is a computable sequence of computable numbers and $S_N f$ converges effectively to f in the L^p-norm as N tends to infinity.

- The Shannon sampling series provides a remarkably simple algorithm to construct a computable sequence of elementary computable functions in $\mathcal{C}\mathcal{B}_\pi^p$ that converges effectively to f.

Behavior for $p = 1$ and $p = \infty$

- For $p = 1$ and $p = \infty$ the Shannon sampling series cannot be used for this purpose.

Theorem

There exists a signal $f_3 \in \mathcal{CB}^1_\pi$ *such that* $S_1 f_3 \notin \mathcal{CB}^1_\pi$, *because* $S_1 f_3 \notin \mathcal{B}^1_\pi$.

Theorem

There exists a signal $f_4 \in \mathcal{CB}^\infty_{\pi,0}$ *such that* $\{S_N f_4\}_{N\in\mathbb{N}}$ *does not converge effectively to* f_4 *in the* L^∞*-norm.*

Conclusions

- We studied the effective (i.e., computable) approximation of bandlimited signals ($\rightarrow$ algorithmic control of the approximation error).
- We gave a necessary and sufficient condition for computability.
- For $p \in (1, \infty) \cap \mathbb{R}_c$ we have:
  1) f computable $\Leftrightarrow$ $f|_{\mathbb{Z}}$ computable,
  2) Shannon sampling series provides a simple algorithm for the effective approximation of f.
- For $p = 1$ and $p = \infty$ we have no correspondence.

Thank you!