Effective Approximation of Bandlimited Signals and Their Samples

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Motivation

Shannon’s sampling theorem links the continuous-time and discrete-time worlds.

- When applying sampling or interpolation, many properties and characteristics of the signal carry over from one domain into the other (e.g., energy in discrete-time = energy in continuous-time).
- We analyze if and how this transition affects the computability of the signal.
In many applications digital hardware is used (CPUs, FPGAs, etc.).

Computability of a signal is directly linked to the approximation with “simple” signals, where we have an “effective”/algorithmic control of the approximation error.

If a signal is not computable, we cannot control the approximation error.
Overview of the Results

We study bandlimited signals \( f \in \mathcal{B}_\pi^p \) with finite \( L^p \)-norm.

Computability continuous-time \( \Leftrightarrow \) computability discrete-time

<table>
<thead>
<tr>
<th>( p \in (1, \infty) )</th>
<th>( p = 1 ) or ( p = \infty )</th>
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<tbody>
<tr>
<td>✓ Correspondence</td>
<td>( \times ) No correspondence</td>
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<tr>
<td>Algorithm: Shannon sampling series</td>
<td>No algorithm exists</td>
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<tr>
<td>Control of the approximation error</td>
<td>No control of the approximation error</td>
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</table>
Turing Machine:
Abstract device that manipulates symbols on a strip of tape according to certain rules.

- Turing machines are an idealized computing model.
- No limitations on computing time or memory, no computation errors.
- Although the concept is very simple, Turing machines are capable of simulating any given algorithm.

Turing machines are suited to study the limitations of a digital computer:

Anything that is not Turing computable cannot be computed on a real digital computer, regardless how powerful it may be.

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Notation

- $c_0$: space of all sequences that vanish at infinity
- $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$: spaces of $p$-th power summable sequences $x = \{x(k)\}_{k \in \mathbb{Z}}$
  
  \[ \|x\|_{\ell^p} = \left( \sum_{k=-\infty}^{\infty} |x(k)|^p \right)^{1/p} \]
- $L^p(\Omega)$, $1 \leq p < \infty$: space of all measurable, $p$th-power Lebesgue integrable functions on $\Omega$
  
  \[ \|f\|_p = \left( \int_{\Omega} |f(t)|^p \, dt \right)^{1/p} \]
- $L^\infty(\Omega)$: space of all functions for which the essential supremum norm $\| \cdot \|_\infty$ is finite
Bandlimited Functions

Definition (Bernstein Space)

Let $\mathcal{B}_\sigma$ be the set of all entire functions $f$ with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leq C(\epsilon) \exp\left((\sigma + \epsilon)|z|\right)$ for all $z \in \mathbb{C}$.

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. The norm for $\mathcal{B}_\sigma^p$ is given by the $L^p$-norm on the real line.

- A function in $\mathcal{B}_\sigma^p$ is called bandlimited to $\sigma$.
- We have $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^r$ for all $1 \leq p \leq r \leq \infty$.
- $\mathcal{B}_\sigma^\infty_{,0}$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.
- $\mathcal{B}_\sigma^2$: space of bandlimited functions with finite energy.
A sequence of rational numbers \( \{ r_n \}_{n \in \mathbb{N}} \) is called **computable sequence** if there exist recursive functions \( a, b, s \) from \( \mathbb{N} \) to \( \mathbb{N} \) such that \( b(n) \neq 0 \) for all \( n \in \mathbb{N} \) and

\[
 r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \quad n \in \mathbb{N}.
\]

- A **recursive function** is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.
Computable Real Numbers

First example of an effective approximation

A real number $x$ is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n \in \mathbb{N}}$ and a recursive function $\xi : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|x - r_n| < 2^{-M}$$

for all $n \geq \xi(M)$.

- $\mathbb{R}_c$: set of computable real numbers
- $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable.
- Commonly used constants like $e$ and $\pi$ are computable.
Computability in $\ell^p$

Computability in Banach spaces: Effective approximation by “simple” elements

A sequence $x = \{x(k)\}_{k \in \mathbb{Z}}$ in $\ell^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, is called computable in $\ell^p$ if every number $x(k)$, $k \in \mathbb{Z}$, is computable and there exist a computable sequence $\{y_n\}_{n \in \mathbb{N}} \subset \ell^p$, where each $y_n$ has only finitely many non-zero elements, all of which are computable as real numbers, and a recursive function $\xi : \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have

$$\|x - y_n\|_{\ell^p} \leq 2^{-M}$$

for all $n \geq \xi(M)$.

• Effective approximation by simple / finite-length sequences
• $\mathcal{C}_{\ell^p}$: set of all sequences that are computable in $\ell^p$
• $\mathcal{C}_{c_0}$: set of all sequences that are computable in $c_0$
We call a function $f$ **elementary computable** if there exists a natural number $L$ and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^L$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$  

- Every elementary computable function is **Turing computable**.
- For every elementary computable function $f$, the norm $\|f\|_{B_{\pi}^p}$ is **computable**.
A function in $f \in \mathcal{B}^p_\pi$, $1 \leq p < \infty$, is called computable in $\mathcal{B}^p_\pi$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n \in \mathbb{N}}$ and a recursive function $\xi: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have
$$
\|f - f_n\|_p \leq 2^{-M}
$$
for all $n \geq \xi(M)$.

- $\mathcal{CB}^p_\pi$: set of all functions that are computable in $\mathcal{B}^p_\pi$.
- $\mathcal{CB}^\infty_\pi,0$: set of all functions that are computable in $\mathcal{B}^\infty_\pi,0$ (analog definition).
A function in $f \in \mathcal{B}_\pi^p$, $1 \leq p < \infty$, is called computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n \in \mathbb{N}}$ and a recursive function $\xi : \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_p \leq 2^{-M}$$

for all $n \geq \xi(M)$.

- $\mathcal{CB}_\pi^p$: set of all functions that are computable in $\mathcal{B}_\pi^p$.
- $\mathcal{CB}_\pi^{\infty,0}$: set of all functions that are computable in $\mathcal{B}_\pi^{\infty,0}$ (analog definition).

We can approximate every function $f \in \mathcal{CB}_\pi^p$ by an elementary computable function, where we have an effective control of the approximation error.
Computable Bandlimited Functions III

For \( f \in \mathcal{CB}_\pi^p \), \( p \in [1, \infty) \cap \mathbb{R}_c \) and all \( M \in \mathbb{N} \) we have

\[
\| f - f_n \|_\infty \leq (1 + \pi) \| f - f_n \|_p \leq \frac{1 + \pi}{2^M}
\]

for all \( n \geq \xi(M) \).

We can approximate any function \( f \in \mathcal{CB}_\pi^p \) by an elementary computable function, where we have an effective and uniform control of the approximation error.
Observation

Let \( f \in CB^p_{\pi}, \ p \in [1, \infty) \cap \mathbb{R}_c \), or \( f \in CB^\infty_{\pi,0} \). Then \( f|_Z = \{f(k)\}_{k \in \mathbb{N}} \) is a computable sequence of computable numbers. Further we have \( f|_Z \in C^{\ell p} \) if \( p \in [1, \infty) \cap \mathbb{R}_c \), and \( f|_Z \in Cc_0 \) if \( p = \infty \).

Continuous-time signal \( f \) computable \( \Rightarrow \) Discrete-time signal \( f|_Z \) computable
Two Questions

Given a computable discrete-time signal, is the corresponding continuous-time signal computable?

Question 1: Is there a simple necessary and sufficient condition for characterizing the computability of $f$?

Question 2: Is there a simple canonical algorithm to actually compute $f$ from the samples $f|_Z$?
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Two Questions

Given a computable discrete-time signal, is the corresponding continuous-time signal computable?

Continuous-time signal $f$ computable × Discrete-time signal $f|_Z$ computable

Question 1:
Is there a simple necessary and sufficient condition for characterizing the computability of $f$?

Question 2:
Is there a simple canonical algorithm to actually compute $f$ from the samples $f|_Z$?
A Necessary and Sufficient Condition

**Theorem**

Let $f \in \mathcal{B}_\pi^p$, $p \in (1, \infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if $f|_\mathbb{Z} \in \mathcal{C}_p$.

- For $p \in (1, \infty) \cap \mathbb{R}_c$, the computability of the discrete-time signal implies the computability of the continuous-time signal.
- This answers Question 1.
A Necessary and Sufficient Condition

Theorem

Let \( f \in \mathcal{B}_\pi^p, \ p \in (1, \infty) \cap \mathbb{R}_c \). Then we have \( f \in \mathcal{C}\mathcal{B}_\pi^p \) if and only if \( f|_\mathbb{Z} \in \mathcal{C}\ell^p \).

- For \( p \in (1, \infty) \cap \mathbb{R}_c \) we have: \( f \) computable \( \iff \) \( f|_\mathbb{Z} \) computable.

That is we have a correspondence between the computable discrete-time signals in \( \mathcal{C}\ell^p \) and the computable continuous-time signals in \( \mathcal{C}\mathcal{B}_\pi^p \).
For \( p = 1 \) we do not have the correspondence. There exist signals that are in \( \mathcal{C}\ell^1 \) (computable in discrete-time), where the corresponding continuous-time signal is not in \( \mathcal{C}\mathcal{B}_1^1_{\pi} \).

**Example:**

- \( f_1(t) = \sin(\pi t)/(\pi t), \ t \in \mathbb{R} \).
- \( f_1 \) is a function of exponential type at most \( \pi \) and we have \( f_1|_{\mathbb{Z}} \in \mathcal{C}\ell^1 \).
- However, \( f_1 \not\in \mathcal{C}\mathcal{B}_1^1_{\pi} \), because \( f_1 \not\in \mathcal{B}_1^1_{\pi} \).
No correspondence for $p = \infty$

For $p = \infty$ we also do not have the correspondence.

Theorem

There exists a $f_2 \in B_{\pi,0}^\infty$ such that $f_2|_\mathbb{Z} \in C_0$ and $f_2 \not\in CB_{\pi,0}^\infty$.

(We even have $f_2(t) \not\in C_c$ for all $t \in \mathbb{R}_c \setminus \mathbb{Z}$).
A Further Necessary and Sufficient Condition

Theorem

Let $f \in \mathcal{B}_p^\pi$, $p \in (1, \infty) \cap \mathbb{R}_c$. We have $f \in \mathcal{CB}_p^\pi$ if and only if

1. $f|_Z$ is a computable sequence of computable numbers,
2. $\|f|_Z\|_{\ell^p} \in \mathbb{R}_c$.

- We do not require that the sequence $f|_Z$ is computable in $\ell^p$, but only that the number $\|f|_Z\|_{\ell^p}$ is computable.
An Answer to Question 2

Shannon sampling series

\[ (S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t - k))}{\pi(t - k)}, \quad t \in \mathbb{R}. \]

Theorem

Let \( p \in (1, \infty) \cap \mathbb{R}_c \) and \( f \in \mathcal{B}_\pi^p \). Then we have \( f \in \mathcal{CB}_\pi^p \) if and only if \( f|_\mathbb{Z} \) is a computable sequence of computable numbers and \( S_N f \) converges effectively to \( f \) in the \( L^p \)-norm as \( N \) tends to infinity.

- The Shannon sampling series provides a remarkably simple algorithm to construct a computable sequence of elementary computable functions in \( \mathcal{CB}_\pi^p \) that converges effectively to \( f \).
Behavior for \( p = 1 \) and \( p = \infty \)

- For \( p = 1 \) and \( p = \infty \) the Shannon sampling series cannot be used for this purpose.

**Theorem**

There exists a signal \( f_3 \in \mathcal{CB}^1_{\pi} \) such that \( S_1 f_3 \notin \mathcal{CB}^1_{\pi} \), because \( S_1 f_3 \notin B^1_{\pi} \).

**Theorem**

There exists a signal \( f_4 \in \mathcal{CB}^\infty_{\pi,0} \) such that \( \{S_N f_4\}_{N \in \mathbb{N}} \) does not converge effectively to \( f_4 \) in the \( L^\infty \)-norm.
Conclusions

• We studied the **effective** (i.e., computable) **approximation** of bandlimited signals (∴ algorithmic control of the approximation error).

• We gave a **necessary and sufficient condition** for computability.

• For $p \in (1, \infty) \cap \mathbb{R}_c$ we have:
  1) $f$ computable $\iff f|_{\mathbb{Z}}$ computable,
  2) Shannon sampling series provides a simple algorithm for the effective approximation of $f$.

• For $p = 1$ and $p = \infty$ we have no correspondence.
Thank you!