Time-Domain Concentration and Approximation of Computable Bandlimited Signals

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Motivation

- Bandlimited signals play a crucial role in signal processing.
- Bandlimited signals have a perfect concentration in the frequency domain.
- However, they cannot simultaneously be perfectly concentrated in the time domain.

We study the time concentration behavior of bandlimited signals from a computational point of view.

Why Study Questions of Computability?

- In many applications digital hardware is used (CPUs, FPGAs, DSPs, etc.).
- Computability of a signal is directly linked to the approximation with “simple” signals, where we have an “effective”/algorithmic control of the approximation error.
- If a signal is not computable, we cannot control the approximation error.

Turing Machine

Turing Machine:

Abstract device that manipulates symbols on a strip of tape according to certain rules.

- Turing machines are an idealized computing model.
- No limitations on computing time or memory, no computation errors.
- Although the concept is very simple, Turing machines are capable of simulating any given algorithm.

Turing machines are suited to study the limitations of a digital computer:

> Anything that is not Turing computable cannot be computed on a real digital computer, regardless how powerful it may be.

A. M. Turing, “On computable numbers, with an application to the Entscheidungsproblem,” *Proceedings of the London Mathematical Society*, vol. s2-42, no. 1, pp. 230–265, Nov. 1936

A. M. Turing, “On computable numbers, with an application to the Entscheidungsproblem. A correction,” *Proceedings of the London Mathematical Society*, vol. s2-43, no. 1, pp. 544–546, Jan. 1937

Notation

- $L^p(\Omega)$, $1 \leqslant p < \infty$: space of all measurable, pth-power Lebesgue integrable functions on Ω
  Norm: $\|f\|_p = \left(\int_\Omega |f(t)|^p \, dt\right)^{1/p}$.
- $L^\infty(\Omega)$: space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite.

Bandlimited Functions

- We consider the Bernstein spaces $\mathcal{B}_\pi^p$: bandlimited signals with finite L^p-norm as characteristic time domain behavior.

Definition (Bernstein Space)

Let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leqslant C(\epsilon)\exp\big((\sigma+\epsilon)|z|\big)$ for all $z \in \mathbb{C}$.

The Bernstein space $\mathcal{B}_\sigma^p$ consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line.

- A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ.
- $\mathcal{B}_\sigma^2$: space of bandlimited functions with finite energy.
- $\mathcal{B}_{\sigma,0}^\infty$: space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish at infinity.
- We have $\mathcal{B}_\sigma^r \subsetneq \mathcal{B}_\sigma^s \subsetneq \mathcal{B}_{\sigma,0}^\infty$ for all $1 \leqslant r < s < \infty$.

Computable Sequences of Rationals

A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}, \qquad n \in \mathbb{N}.$$

- A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions. Recursive functions are computable by a Turing machine.

Computable Real Numbers

First example of an effective approximation

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$|x - r_n| < 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathbb{R}_c$: set of computable real numbers
- $\mathbb{R}_c$ is a field, i.e., finite sums, differences, products, and quotients of computable numbers are computable.
- Commonly used constants like e and π are computable.

Elementary Computable Functions

We call a function f elementary computable if there exists a natural number L and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

- Every elementary computable function is Turing computable.
- For every elementary computable function f, the norm $\|f\|_{\mathcal{B}_\pi^p}$ is computable.

Computable Bandlimited Signals: Definition A

Definition A

A function in $f \in \mathcal{B}_\pi^p$, $1 \leqslant p < \infty$, is called computable in $\mathcal{B}_\pi^p$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n \in \mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_p \leqslant 2^{-M}$$

for all $n \geqslant \xi(M)$.

- $\mathcal{C}\mathcal{B}_\pi^p$: set of all signals that are computable in $\mathcal{B}_\pi^p$.
- $\mathcal{C}\mathcal{B}_{\pi,0}^\infty$: set of all signals that are computable in $\mathcal{B}_{\pi,0}^\infty$ (analogous definition).

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- $\mathcal{CB}_\pi^p$: set of all signals that are computable in $\mathcal{B}_\pi^p$.
- $\mathcal{CB}_{\pi,0}^\infty$: set of all signals that are computable in $\mathcal{B}_{\pi,0}^\infty$ (analogous definition).

We can approximate every signal $f \in \mathcal{CB}_\pi^p$ by an elementary computable function, where we have an effective control of the approximation error.

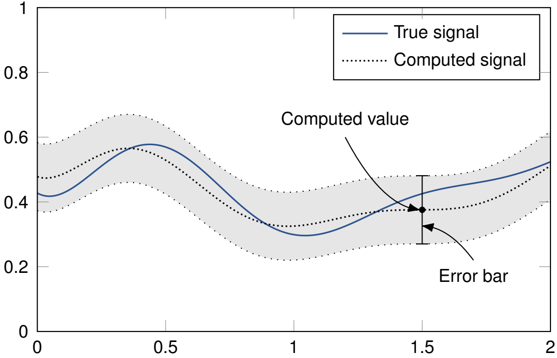
Effective Approximation

For $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ and all $M \in \mathbb{N}$ we have

$$\|f - f_n\|_\infty \leqslant (1+\pi)\|f - f_n\|_p \leqslant \frac{1+\pi}{2^M}$$

for all $n \geqslant \xi(M)$.

We can approximate any signal $f \in \mathcal{CB}_\pi^p$ by an elementary computable function, where we have an effective and uniform control of the approximation error.



Advantages / Drawbacks of Definition A

Advantages:

- Intuitively clear
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Drawbacks:

- Difficult to answer questions about the time concentration behavior
- Connection to the usual definition of a computable continuous function unclear

Computable Continuous Functions

Definition (Computable Continuous Function)

A function $f\colon \mathbb{R} \to \mathbb{R}$ is a called computable continuous function if

1. f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of real numbers.
2. there exists a recursive function $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $L, M \in \mathbb{N}$ we have: $|t_1 - t_2| \leqslant 1/d(L, M)$ implies $|f(t_1) - f(t_2)| \leqslant 2^{-M}$ for all $t_1, t_2 \in [-L, L]$.

Time Concentration

"Amount" of the signal f in $[-L, L]$: $$\int_{-L}^{L} |f(t)|^p \, dt$$

Time concentration on $[-L, L]$: $$\int_{-\infty}^{\infty} |f(t)|^p \, dt - \int_{-L}^{L} |f(t)|^p \, dt = \int_{|t|>L} |f(t)|^p \, dt$$

- The smaller this value, the more concentrated is the signal.
- When is the convergence effective?

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Observation: If $f \in \mathcal{CB}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, then

- $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$.
- Since $\{\int_{|t|\leqslant L} |f(t)|^p \, dt\}_{L\in\mathbb{N}}$ is monotonically increasing, the convergence is effective.
- For $f \in \mathcal{CB}_\pi^p$ we have an algorithmic description of the time concentration behavior.

Computable Bandlimited Signals: Definition B

Definition of a computable bandlimited signal using the idea of effective time concentration:

Definition B

We say that a signal $f \in \mathcal{B}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$ has an effectively computable time concentration if

1. f is a computable continuous function, and
2. there exists a recursive function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ we have

$$\left| \|f\|_{\mathcal{B}_\pi^p}^p - \int_{-L}^{L} |f(t)|^p \, dt \right| \leqslant \frac{1}{2^M}$$

for all $L \geqslant \xi(M)$.

$\mathcal{CT}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$: set of such signals.

For $p = \infty$, i.e., signals $f \in \mathcal{B}_{\pi,0}^\infty$, we use an analogous definition, with $|\|f\|_{\mathcal{B}_{\pi,0}^\infty} - \max_{|t| \leqslant L} |f(t)| \, dt| \leqslant 1/2^M$.

Main Results

Theorem 1

Let $p \in (1, \infty) \cap \mathbb{R}_c$. Then we have $\mathcal{CB}_\pi^p = \mathcal{CT}_\pi^p$.

- For $p \in (1, \infty) \cap \mathbb{R}_c$, the sets $\mathcal{CB}_\pi^p$ and $\mathcal{CT}_\pi^p$ (Definitions A and B) coincide.
- No longer true for $p = 1$ and $p = \infty$.

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Theorem 2

Let $p \in (1, \infty) \cap \mathbb{R}_c$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if $f \in \mathcal{B}_\pi^p$, f is a computable continuous function, and $\|f\|_{\mathcal{B}_\pi^p} \in \mathbb{R}_c$.

- Simple characterization of $\mathcal{CB}_\pi^p$ signals.
- No longer true for $p = 1$ and $p = \infty$.

Conclusions

- We studied the time concentration behavior of bandlimited signals from a computational point of view.
- We introduced a definition of a computable bandlimited signal based on the notion of effective time concentration (Definition B).
- We showed that Definition B is equivalent to Definition A (for $p \in (1, \infty) \cap \mathbb{R}_c$).
- Connections to computable continuous functions are revealed.
- Our findings lead to a simple characterization of computable bandlimited signals.

Thank you!