Solvability of the PAPR Problem for OFDM with Reduced Compensation Set

Holger Boche and Ullrich J. Mönich

Technische Universität München
Lehrstuhl für Theoretische Informationstechnik

June 21, 2018
2018 IEEE International Symposium on Information Theory

Orthogonal Transmission Scheme

In modern communication systems, orthogonal transmission schemes, e.g., orthogonal frequency division multiplexing (OFDM), are widely used.

Transmit Signal: $$s(t) = \sum_{k \in \mathcal{I}} c_k \phi_k(t), \quad t \in [t_1, t_2],$$

- $\{\phi_k\}_{k \in \mathcal{I}}$ is an orthonormal system (ONS) of functions.
- Each ϕ_k is called carrier.
- $\{c_k\}_{k \in \mathcal{I}} \subset \mathbb{C}$ are the information bearing coefficients.
- $t_2 - t_1$ is the signal duration.

Orthogonal Transmission Scheme

- Orthogonal transmission schemes have many desirable properties (e.g. high data rates, simple equalization).
- But: Large peak-to-average power ratios (PAPRs) are a problem.

The Peak To Average Power Ratio (PAPR)

For a signal $s \in L^2[-\pi, \pi]$, we define

$$\text{PAPR}(s) = \frac{\|s\|_{L^\infty[-\pi,\pi]}}{\|s\|_{L^2[-\pi,\pi]}},$$

(The PAPR is usually defined as the square of this value $\rightarrow$ not important here.)

The PAPR for an Orthogonal Transmission Scheme:

For an ONS $\{\phi_k\}_{k\in\mathcal{I}} \subset L^2[-\pi, \pi]$, the PAPR of the transmit signal

$$s(t) = \sum_{k\in\mathcal{I}} c_k \phi_k(t), \quad t \in [-\pi, \pi],$$

is given by

$$\text{PAPR}(s) = \frac{\left\|\sum_{k\in\mathcal{I}} c_k \phi_k\right\|_{L^\infty[-\pi,\pi]}}{\|c\|_{\ell^2(\mathcal{I})}}$$

because $\|s\|_{L^2[-\pi,\pi]} = \|c\|_{\ell^2(\mathcal{I})}$.

Problematic PAPR Behavior

General Result on PAPR Behavior

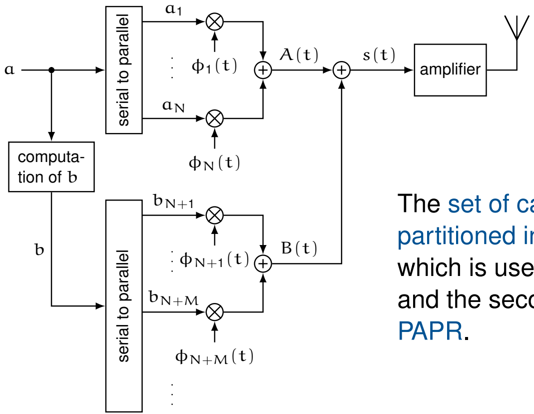
Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[-\pi,\pi]$, there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that

$$\left\| \sum_{n=1}^{N} c_n \phi_n \right\|_{L^\infty[-\pi,\pi]} \geqslant \sqrt{N}.$$

$\Rightarrow$ PAPR control is always necessary for large numbers of carriers N.

Tone Reservation

A popular method to fight large PAPRs is tone reservation.



The set of carriers $\{\phi_k\}_{k\in\mathcal{I}}$ is partitioned into two sets, the first of which is used to carry the information, and the second of which to reduce the PAPR.

Tone Reservation

Let $\{\phi_k\}_{k\in\mathcal{I}} \subset L^2[-\pi,\pi]$ be an ONS. We assume that $\|\phi_k\|_\infty < \infty$, $k \in \mathcal{I}$.

The index set $\mathcal{I}$ is partitioned into two disjoint sets $\mathcal{K}$ and $\mathcal{K}_{\mathcal{I}}^{\complement}$ (finite or infinite).

Index set $\mathcal{I}$ → $\mathcal{K} \longrightarrow \{\phi_k\}_{k\in\mathcal{K}}$ (Information)

Index set $\mathcal{I}$ → $\mathcal{K}_{\mathcal{I}}^{\complement} \longrightarrow \{\phi_k\}_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}}$ (PAPR reduction)

Goal: For a given $a \in \ell^2(\mathcal{K})$, find $b \in \ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})$ such that the peak value of

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}} b_k\phi_k(t)}_{=:B(t)}, \quad t \in [-\pi,\pi],$$

is as small as possible.

- $A(t)$: signal part which contains the information (information signal).
- $B(t)$: signal part which is used to reduce the PAPR (compensation signal).

Tone Reservation

- The tone reservation method is easy to define but hard to analyze analytically.
- Questions are:
  - What is the best possible reduction of the PAPR?
  - What information set $\mathcal{K}$ should be used?
  - How to find the optimal compensation sequence $\mathbb{b}$?

Solvability of the PAPR Problem

Definition (Solvability of the PAPR problem)

For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2[-\pi,\pi]$ and a set $\mathcal{K}\subset\mathcal{I}$, we say that the PAPR problem is solvable with finite constant $C_{\mathsf{EX}}^{\mathcal{I}}$, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})$ such that

$$\left\|\sum_{k\in\mathcal{K}} a_k\phi_k+\sum_{k\in\mathcal{K}_{\mathcal{I}}^{\complement}} b_k\phi_k\right\|_{L^\infty[-\pi,\pi]}\leqslant C_{\mathsf{EX}}^{\mathcal{I}}\|a\|_{\ell^2(\mathcal{K})}. \quad (1)$$

Solvability of the PAPR Problem

- If the PAPR reduction problem is solvable, condition (1) immediately implies that $\|b\|_{\ell^2(\mathcal{K}_{\mathcal{I}}^{\complement})} \leqslant C_{\mathrm{EX}}^{\mathcal{I}} \|a\|_{\ell^2(\mathcal{K})}$, because

$$\left(\sum_{k \in \mathcal{K}_{\mathcal{I}}^{\complement}} |b_k|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{k \in \mathcal{K}} |a_k|^2 + \sum_{k \in \mathcal{K}_{\mathcal{I}}^{\complement}} |b_k|^2 \right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k \in \mathcal{K}} a_k \phi_k(t) + \sum_{k \in \mathcal{K}_{\mathcal{I}}^{\complement}} b_k \phi_k(t) \right|^2 dt \right)^{\frac{1}{2}}$$
$$\leqslant \left\| \sum_{k \in \mathcal{K}} a_k \phi_k + \sum_{k \in \mathcal{K}_{\mathcal{I}}^{\complement}} b_k \phi_k \right\|_{L^\infty[-\pi,\pi]}$$

$\rightarrow$ The energy of the compensation signal is bounded by $(C_{\mathrm{EX}}^{\mathcal{I}} \|a\|_{\ell^2(\mathcal{K})})^2$.

- We have $\mathrm{PAPR}(s) \leqslant C_{\mathrm{EX}}^{\mathcal{I}}$.

OFDM

Carriers in OFDM:

- $\{\phi_k\}_{k\in\mathcal{I}} = \{e^{ik\,\cdot}\}_{k\in\mathcal{I}}$ (set of complex exponentials).

Full carrier set ($\mathcal{I} = \mathbb{Z}$):

- If $\mathcal{I} = \mathbb{Z}$ we use the carriers $\{e^{ik\,\cdot}\}_{k\in\mathbb{Z}}$, i.e., positive as well as negative frequencies.
- The set of carriers $\{e^{ik\,\cdot}\}_{k\in\mathbb{Z}}$ is a complete ONS in $L^2[-\pi,\pi]$.
- Fully developed theory [BMT17, BF13].

Reduced carrier set ($\mathcal{I} = \mathbb{N}$):

- In applications the setting $\mathcal{I} = \mathbb{N}$, in which only the positive frequencies are used, is also important.
- The set of carriers $\{e^{ik\,\cdot}\}_{k\in\mathbb{N}}$ is not complete in $L^2[-\pi,\pi]$.

$\Rightarrow$ The previous results cannot be applied.

[BMT17] H. Boche, U. J. Mönich, and E. Tampubolon, "Complete characterization of the solvability of PAPR reduction for OFDM by tone reservation," in *Proceedings of the 2017 IEEE International Symposium on Information Theory*, Jun. 2017, pp. 2023–2027

[BF13] H. Boche and B. Farrell, "On the peak-to-average power ratio reduction problem for orthogonal transmission schemes," *Internet Mathematics*, vol. 9, no. 2–3, pp. 265–296, 2013

OFDM with Reduced Carrier Set

In order to compare the two scenarios ($\mathcal{I} = \mathbb{Z}$ and $\mathcal{I} = \mathbb{N}$), we assume that the information set $\mathcal{K}$ is a subset of $\mathbb{N}$.

As compensation sets we consider:

- **Full compensation set:** $\mathcal{K}_{\mathbb{Z}}^{\complement} = \mathbb{Z} \setminus \mathcal{K}$
- **Reduced compensation set:** $\mathcal{K}_{\mathbb{N}}^{\complement} = \mathbb{N} \setminus \mathcal{K}$

OFDM with Reduced Carrier Set

Definition (Solvability of the OFDM PAPR problem with reduced carrier set)

For a set $\mathcal{K} \subset \mathbb{N}$, we say that the OFDM PAPR problem with reduced carrier set is solvable with finite constant $C_{\mathrm{EX}}^{\mathbb{N}}$, if for all $a \in \ell^2(\mathcal{K})$ there exists a $b \in \ell^2(\mathcal{K}_{\mathbb{N}}^{\complement})$, such that

$$\left\| \sum_{k\in\mathcal{K}} a_k \, e^{ik\,\cdot} + \sum_{k\in\mathcal{K}_{\mathbb{N}}^{\complement}} b_k \, e^{ik\,\cdot} \right\|_{L^\infty[-\pi,\pi]} \leqslant C_{\mathrm{EX}}^{\mathbb{N}} \|a\|_{\ell^2(\mathcal{K})}.$$

Remark

A necessary condition for the solvability of the OFDM PAPR problem with reduced carrier set is the solvability of the OFDM PAPR problem with full carrier set. ($\mathcal{K}_{\mathbb{N}}^{\complement} \subset \mathcal{K}_{\mathbb{Z}}^{\complement}$)

A Characterization of Solvability

$$\mathfrak{F}^1(\mathcal{K}) = \left\{ f \in L^1[-\pi,\pi] \colon f = \sum_{k \in \mathcal{K}} a_k \, e^{ik\,\cdot} \text{ for some } \{a_k\}_{k\in\mathcal{K}} \subset \mathbb{C} \right\},$$

Theorem

Let $\mathcal{K} \subset \mathbb{N}$. The OFDM PAPR problem with reduced carrier set ($\mathcal{I} = \mathbb{N}$) is solvable if and only if there exists a constant C_1 such that

$$\|f\|_{L^2[-\pi,\pi]} \leqslant C_1 \|f\|_{L^1[-\pi,\pi]}$$

for all $f \in \mathfrak{F}^1(\mathcal{K})$.

- The theorem gives a complete characterization of solvability with reduced carrier set.

Comparison to OFDM With Full Carrier Set

If we compare the result for the reduced carrier set with the result for the full carrier set (complete ONS), we get the following corollary.

Corollary

Let $\mathcal{K} \subset \mathbb{N}$. The OFDM PAPR problem with reduced carrier set is solvable if and only if it is solvable with full carrier set.

- With respect to solvability, the usage of the full compensation set $\mathcal{K}_{\mathbb{Z}}^{\mathsf{C}}$ does not give any advantage over the usage of the reduced compensation set $\mathcal{K}_{\mathbb{N}}^{\mathsf{C}}$.
- However, the optimal constants are larger in general.

First Example

Example

Let $\mathcal{K} = \{2^k\}_{k \in \mathbb{N}}$. Then the OFDM PAPR problem is solvable.

- Since $\{2^k\}_{k \in \mathbb{N}}$ is a lacunary sequence, it can be proved that the norm inequality

$$\|f\|_{L^2[-\pi,\pi]} \leqslant C_1 \|f\|_{L^1[-\pi,\pi]}$$

holds for all $f \in \mathfrak{F}^1(\mathcal{K})$.

A Necessary Condition for Solvability

Information sets $\mathcal{K}$ with long arithmetic progressions are bad for the control of the PAPR.

Definition: An arithmetic progression of length k with difference d is a set of the form $\{n, n+d, n+2d, \ldots, n(k-1)d\}$ where $n, d \in \mathbb{N}$.

For $\mathcal{K} \subset \mathbb{N}$, we call

$$\overline{d}(\mathcal{K}) := \limsup_{N \to \infty} \frac{|\mathcal{K} \cap \{0, \ldots, N\}|}{N+1}$$

the upper density of $\mathcal{K}$.
Using Szemerédi's result on arithmetic progressions it has been shown:

Theorem (Boche, Farrell)

Let $\mathcal{K} \subset \mathbb{N}$ be an arbitrary set such that $\overline{d}(\mathcal{K}) > 0$. Then, the OFDM PAPR problem is not solvable.

Necessary condition: $\overline{d}(\mathcal{K}) = 0$ needs to be satisfied, otherwise the PAPR problem cannot be solvable.

Second Example

The condition $\overline{d}(\mathcal{K}) = 0$ is necessary but not sufficient.

Example

Let $\mathcal{K}$ be the set of all primes $\mathbb{P}$. Then we have

$$\overline{d}(\mathbb{P}) = \limsup_{N \to \infty} \frac{|\mathbb{P} \cap \{0, \dots, N\}|}{N+1} = 0.$$

However, the set of primes $\mathbb{P}$ contains arbitrarily long arithmetic progressions (deep result by Green and Tao 2008).
$\Rightarrow$ The OFDM PAPR problem is not solvable.

B. Green and T. Tao, "The primes contain arbitrarily long arithmetic progressions," *Annals of Mathematics*, vol. 167, no. 2, pp. 481–547, 2008

Size of the Set of Information Sequences for which the PAPR Problem is not Solvable

Question: If the PAPR is not solvable, how many information sequences $a \in \ell^2(\mathcal{K})$ are there for which we cannot control the PAPR?

Theorem

Let $\mathcal{K} \subset \mathbb{N}$ such that the PAPR problem is not solvable. Then

$$\{a \in \ell^2(\mathcal{K}) \colon \text{PAPR cannot be controlled}\}$$

is a residual set.

If the PAPR is not solvable, then the set of sequences $a \in \ell^2(\mathcal{K})$ for which the peak value is infinite is a residual set, i.e., large in a topological sense.

Conclusions

- We have analytically studied the tone reservation method for OFDM and a reduced carrier set (only positive frequencies)
- We gave a full characterization of the information sets for which the PAPR problem is solvable.
- The reduction of the carrier set does not affect the solvability.
- The optimal constants are worse in general.

Thank you!